

The L^p Dirichlet Problem for Divergence Form Elliptic Operators with Non-smooth Coefficients

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We present conditions on the disagreement between two elliptic operators which ensure the preservation of L^p solvability for the Dirichlet problem on the unit ball in \mathbb{R}^n . The conditions depend on p and differ from previous results on this question. Furthermore, the conditions are sharp in the sense that they are equivalent to L^p

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1. INTRODUCTION

In this paper, our focus is the L^p Dirichlet problem for elliptic divergence form operators with bounded measurable coefficients. Given two such operators L_0 and L_1 , we have formulated a condition (depending on p) on the difference between L_0 and L_1 such that if L_0 is solvable for boundary data in L^p , and the condition holds, then L_1 is also solvable for data in L^p . Furthermore, this condition is sharp in the sense that no stronger conclusion can be drawn from the hypotheses.

The issue of preservation of solvability for the L^p Dirichlet problem has been the setting for much work (cf. [5, 8, 10–12]), and the new results presented here owe much to ideas from the above works, and especially from [12]. To describe these results, we first recall the Dirichlet problem and its connection to the theory of weights.

Consider operators $L = \operatorname{div} A \nabla$ where $A = A(X)$ is a real, symmetric matrix of bounded measurable functions, and furthermore, A is uniformly elliptic; that is, there exist positive constants λ_1 and λ_2 such that for any $\xi \in \mathbb{R}^n$,

$$\lambda_1 |\xi|^2 \leq \langle A(X) \xi, \xi \rangle \leq \lambda_2 |\xi|^2.$$

Then given a domain $D \subseteq \mathbb{R}^n$ and $g \in L^p$, the L^p Dirichlet problem asks for a function u such that

$$\begin{aligned} Lu &= 0 && \text{in } D \\ u|_{\partial D} &= g \in L^p. \end{aligned} \tag{1.1}$$

For our discussion, we will let $D = B(0, 1) = B_1$, the unit ball centered at 0.

Solvability of the classical Dirichlet problem for L in B_1 [20] gives rise to the mutually absolutely continuous L -harmonic or elliptic measures ω_L^X on ∂B_1 , where $u(X) = \int_{\partial B_1} g(Q) d\omega_L^X(Q)$. We will call $\omega_L = \omega_L^0$ the elliptic measure for L in B_1 . Furthermore, these measures ω_L are doubling: if for $Q \in \partial B_1$ we define $B(Q, r) = \{X \in \mathbb{R}^n : |X - Q| < r\}$ and $\Delta_r(Q) = B(Q, r) \cap \partial B_1$, then there exists a constant C such that for any $\Delta_r(Q) \subseteq \partial B_1$,

$$\omega_L(\Delta_{2r}(Q)) \leq C \omega_L(\Delta_r(Q)). \quad (1.2)$$

The correspondence between operators L and the measures ω_L leads to important connections to the theory of weights. Letting $d\sigma$ denote surface measure on ∂B_1 , and given a weight $k \geq 0$ on ∂B_1 such that $k d\sigma$ is doubling, recall that k is in the class $B_p(d\sigma) = B_p$ ($p > 1$) if there exists a constant C such that for all $\Delta_r \subseteq \partial B_1$,

$$\left(\frac{1}{\sigma(\Delta_r)} \int_{\Delta_r} k^p d\sigma \right)^{1/p} \leq C \frac{1}{\sigma(\Delta_r)} \int_{\Delta_r} k d\sigma.$$

We also note that the class $A_\infty(d\sigma)$ can be described by $A_\infty = \bigcup_{p > 1} B_p(d\sigma)$ (cf. [4]).

From work in [3] together with results of [22], the Dirichlet problem is solvable in L^p for L if and only if the corresponding weight $k_L = d\omega_L/d\sigma$ is in $B_{p'}(d\sigma)$, where $1/p + 1/p' = 1$. We now discuss some previous results on preservation on B_p class which lead to our new results.

Fabes, Jerison and Kenig [8] first obtained a preservation result, giving a solvability condition when the matrix of coefficients is continuous. Their method was to regard such an operator L as a perturbation of an operator with radially independent coefficients, which are known to be solvable in L^2 ([17]).

In 1986, Dahlberg obtained results on this question for nonsmooth operators. Given two operators $L_0 = \operatorname{div} A_0 \nabla$ and $L_1 = \operatorname{div} A_1 \nabla$, and defining their disagreement function to be $a(X) = \sup_{Y \in B(X, \delta(X)/2)} |A_0(Y) - A_1(Y)|$, where $\delta(X)$ is the distance of X to ∂B_1 , Dahlberg showed the following.

THEOREM 1.3 [5]. *Let $\Delta = \Delta(Q, r) \subseteq \partial B_1$, and define $T(\Delta) = \{Y \in B_1 : |Y - Q| \leq r\}$, called the Carleson region over Δ . Let*

$$h(r) = \left(\sup_{Q \in \partial B_1} \int_{T(\Delta)} \frac{a^2(X)}{\delta(X)} \frac{dX}{\sigma(\Delta)} \right). \quad (1.3.1)$$

If $\lim_{r \rightarrow 0} h(r) = 0$, then for any q such that $k_0 \in B_q(d\sigma)$, $k_1 \in B_q(d\sigma)$ also.

Expression (1.3.1) is the "Carleson norm" for the measure $(a^2(X)/\delta(X)) dX$. Dahlberg's result leads to two natural questions. First, if we assume merely that the expression in (1.3.1) is bounded, can we still make a useful conclusion? Second, we note that Dahlberg's condition preserves *all* q for which $k_0 \in B_q(d\sigma)$. Is there a condition depending on q that preserves B_q for that particular q ?

The first question has been answered by R. Fefferman, Kenig, and Pipher ([12]):

THEOREM 1.4 [12]. *If $(a^2(X)/\delta(X)) dX$ is a Carleson measure of finite norm, and $k_0 \in A_\infty(d\sigma)$, then $k_1 \in A_\infty(d\sigma)$.*

Since $A_\infty = \bigcup_{p>1} B_p$, Theorem 1.4 says that if $(a^2(X)/\delta(X)) dX$ is Carleson and we know L_0 is solvable in L^p , then L_1 is solvable in L^q for *some* q , which may be different from p . Hence a weaker hypothesis than in Theorem 1.3 leads to an interesting conclusion, but still leaves the second question to be answered in this paper.

Our main result is twofold. First, we give a condition that ensures that if the $L^{p'}$ Dirichlet problem is solvable for L_0 , with $p \geq 2$, (where $1/p + 1/p' = 1$) then it is solvable for L_1 . This condition is the following integral inequality, which must hold for every surface ball Δ :

$$\left[\int_{T(\Delta)} \frac{a^2(X)}{\delta(X)} \left(\frac{G_1(X)}{\delta(X)} \right)^p \frac{dX}{\sigma(\Delta)} \right]^{1/p} \leq C \frac{\omega_1(\Delta)}{\sigma(\Delta)} \quad (S_p)$$

where C is a fixed constant independent of Δ . Here $G_1(X)$ denotes the Green's function for L_1 on B_1 , evaluated at the point 0. We will refer to the above condition as (S_p) , the solvability condition for the $L^{p'}$ Dirichlet problem.

The second part of the main result is a more general solvability condition which preserves solvability when the boundary data is in $L^{p'}$ with $p > 1$. This alternative condition, which can be thought of as a quadratic version of (S_p) , reads:

$$\left\{ \int_{Q \in \mathcal{A}_r} \left[\int_{\Gamma_r(Q)} \frac{a^2(X)}{\delta^n(X)} \left[\frac{G_1(X)}{\delta(X)} \right]^2 dX \right]^{p/2} \frac{d\sigma(Q)}{\sigma(\Delta)} \right\}^{1/p} \leq C \frac{\omega_1(\Delta)}{\sigma(\Delta)} \quad (QS_p)$$

where $\Gamma_r(Q)$ is a cone over Q , truncated at height r .

We will show that the two conditions (S_p) and (QS_p) are in fact equivalent when $p \geq 2$. The advantage to formulating (S_p) in addition to (QS_p) is that from (S_p) , we will show that $(a^2(X)/\delta(X)) dX$ is in fact a Carleson measure. Furthermore, the condition (S_p) will be used to prove the sharpness of our results.

In Section 2 below, we review necessary background. Sections 3 and 4 present our main results involving (QS_p) and (S_p) . Section 5 demonstrates the equivalence of the two conditions, and Section 6 establishes the sharpness of these results.

We wish to point out that a recent, independent result of R. Fefferman [11] also gives a criterion for preservation of L^p solvability of the Dirichlet problem. This criterion, while dependent on p , is substantially different from the condition presented here, but is certainly important as an additional answer to the question treated here.

2. BACKGROUND

We first establish notation to be used throughout our discussion. For $X \in \mathbb{R}^n$, $r > 0$, let $B(X, r) = \{Y \in \mathbb{R}^n : |X - Y| < r\}$, and let $B_1 = B(0, 1)$. Q will denote a point on the boundary ∂B_1 , and $\Delta(Q, r) = \Delta_r = B(Q, r) \cap \partial B_1$, a surface ball of radius $r = \text{rad}(\Delta_r)$. Given a surface ball $\Delta = \Delta(Q, r)$, let $T(\Delta) = B(Q, r) \cap B_1$, called the Carleson region associated to Δ . For $X \in B_1$, we let $\delta(X)$ be the distance from X to ∂B_1 , and let $X^* = X/|X|$, i.e., the projection of X onto ∂B_1 . Also, let Δ_X denote the surface ball $\Delta(X^*, \delta(X))$.

Unless otherwise specified, C will denote a constant, not necessarily the same at each occurrence, which is independent of everything except perhaps the dimension n , the ellipticity constants λ_1 and λ_2 , and possibly p , when we are working with a particular B_p or L^p .

We will also use the relations “ \approx ” and “ \lesssim .” By “ $f \lesssim g$ ” we mean that there exists a constant C such that $f \leq Cg$. By “ $f \approx g$ ” we mean that $f \lesssim g$ and $g \lesssim f$.

We now review several known results, beginning with Hölder continuity of solutions and Moser’s Harnack inequality. Here the notation \Subset denotes compact containment.

THEOREM 2.1 [7, 23]. *Given any solution of $Lu = 0$ in $D \subseteq \mathbb{R}^n$, and for any $D'' \Subset D' \Subset D$, there exist constants C and α , dependent only on D'' , D' , and the ellipticity constants, such that for all $X, Y \in D''$,*

$$|u(X) - u(Y)| \leq C \|u\|_{L^2(D')} |X - Y|^\alpha.$$

THEOREM 2.2 [21]. *For any positive solution of $Lu = 0$ in D , if $D' \Subset D$, then*

$$\max_{D'} u \leq C \min_{D'} u,$$

where C depends only on D' , D , and the ellipticity constants.

Caffarelli, Fabes, Mortola and Salsa showed that Moser's theorem can be extended to a statement at the boundary of the domain. We state their results for the ball B_1 , but in fact their work was done in more generality.

THEOREM 2.3 [3]. *Say $u > 0$ is a solution to L in B_1 such that $u = 0$ on Δ_{4r} . Let $A_r \in T(\Delta_r)$ be such that $\delta(A_r) \approx r$. Then there exists a constant C such that*

$$\sup_{X \in T(\Delta)} u(X) \leq C u(A_r).$$

Another result from [3] is the "comparison principle" for positive solutions. This theorem states that if two positive solutions vanish on a surface ball, then they vanish at the same rate near a smaller surface ball.

THEOREM 2.4 [3]. *Say u, v are two positive solutions to L in B_1 , with $u = v = 0$ on $\Delta(Q, r) \subseteq \partial B_1$. Let $A_{r/4} \in T(\Delta(Q, r/4))$ be such that $\delta(A_{r/4}) \approx r/4$. Then*

$$\sup_{X \in T(\Delta(Q, r/4))} \frac{u(X)}{v(X)} \leq C \frac{u(A_{r/4})}{v(A_{r/4})}.$$

The above comparison theorem is related to a version in terms of the Green's function and elliptic measure for L .

THEOREM 2.5 [3]. *Let $G(X, Y)$ be the Green's function for L in B_1 . Then*

$$G(X, Y) \approx \frac{\omega^X(\Delta(Y^*, \delta(Y)))}{\delta^{n-2}(Y)}$$

for all $X \in B_1 \setminus T(\Delta(Y^*, 2\delta(Y)))$.

Other fundamental properties of the Green's function, due to Grüter and Widman, are in [14].

We now recall maximal function operators which play a key role in the study of the Dirichlet problem.

DEFINITION 2.6. Given a measure μ and a function f on ∂B_1 , the Hardy–Littlewood maximal function of f with respect to μ is

$$M_\mu(f)(Q) = \sup_{\Delta \ni Q} \frac{1}{\mu(\Delta)} \int_{x \in \Delta} |f(x)| d\mu(x).$$

DEFINITION 2.7. For a fixed angle $\theta < \pi/2$, for $Q \in \partial B_1$, let $\Gamma(Q)$ denote the interior cone with aperture θ , vertex at Q , and axis along the radial line joining Q to 0. Then the non-tangential maximal function of a function u on B_1 is

$$Nu(Q) = \sup_{X \in \Gamma(Q)} |u(X)|.$$

Also, we define a variant of N as

$$\tilde{N}u(Q) = \sup_{X \in \Gamma(Q)} \left[\int_{B(X, \delta(X)/2)} u^2(Y) \frac{dY}{\delta^n(X)} \right]^{1/2}.$$

Note that by (2.2), Nu and $\tilde{N}u$ are equivalent when u is a solution. We also define the square function, or area integral operator.

DEFINITION 2.8. For u a function on B_1 , $\Gamma(Q)$ a cone, the area integral S of u is given by

$$S^2u(Q) = \int_{X \in \Gamma(Q)} |\nabla u(X)|^2 \delta^{2-n}(X) dX.$$

Note that while ∇u need not exist pointwise, we can still make sense of ∇u in terms of L^2 averages by Cacciopoli's inequality:

THEOREM 2.9.

$$\int_{B(X, \delta(X)/4)} |\nabla u(Y)|^2 dY \leq \frac{C}{\delta^2(X)} \int_{B(X, \delta(X)/2)} |u(Y)|^2 dY.$$

The following relationship of Su to Nu is due to Dahlberg, Jerison and Kenig.

THEOREM 2.10 [6]. *Let L be an elliptic operator with solution u and associated elliptic measure ω . For $p > 1$, and for any positive measure μ which is A_∞ with respect to ω , $\|Nu\|_{L^p(d\mu)} \approx \|Su\|_{L^p(d\mu)}$.*

3. PRESERVATION OF SOLVABILITY FOR THE L^p DIRICHLET PROBLEM

Consider two elliptic operators $L_0 = \operatorname{div} A_0 \nabla$ and $L_1 = \operatorname{div} A_1 \nabla$. We let $\omega_0 = \omega_{L_0}^0$, and likewise for ω_1 , while surface measure on ∂B_1 will be denoted by $d\sigma$, or by absolute value signs $|\cdot|$. $G_0(X)$ and $G_1(X)$ will denote the respective Green's functions for L_0 and L_1 , evaluated at the point 0.

Finally, following Dahlberg, let $\varepsilon(Y) = A_1(Y) - A_0(Y)$, and define the disagreement function

$$a(X) = \sup_{Y \in B(X, \delta(X)/2)} |\varepsilon(Y)|.$$

In Section 1 we introduced the condition (QS_p) and (S_p) . We first state the theorem involving (QS_p) .

THEOREM 3.1. *For $p > 1$, suppose that the Dirichlet problem in $L^{p'}$ is solvable for L_p , where $1/p + 1/p' = 1$. If there exists a constant C such that for every surface ball $\Delta \subseteq \partial B_1$,*

$$\left\{ \int_{Q \in \Delta_r} \left[\int_{T(Q)} \frac{a^2(X)}{\delta^n(X)} \left[\frac{G_1(X)}{\delta(X)} \right]^2 dX \right]^{p/2} \frac{d\sigma(Q)}{\sigma(\Delta)} \right\}^{1/p} \leq C \frac{\omega_1(\Delta)}{\sigma(\Delta)}, \quad (QS_p)$$

then the Dirichlet problem in $L^{p'}$ is solvable for L_1 .

Note that in terms of weight spaces B_p , the above theorem can be restated:

THEOREM 3.1'. *Let $d\omega_0 = k_0 d\sigma$, and likewise for k_1 . For $p > 1$, say that $k_0 \in B_p(d\sigma)$. If the condition (QS_p) holds, then $k_1 \in B_p(d\sigma)$ also.*

Before proving Theorem 3.1, we state the result involving the condition (S_p) . This alternative result, which holds for $p \geq 2$, will be proved in Section 4.

THEOREM 3.2. *For $p \geq 2$, suppose that the Dirichlet problem in $L^{p'}$ is solvable for L_0 , where $1/p + 1/p' = 1$. If there exists a constant C such that for every surface ball $\Delta \subseteq \partial B_1$,*

$$\left[\int_{T(\Delta)} \frac{a^2(X)}{\delta(X)} \left[\frac{G_1(X)}{\delta(X)} \right]^p \frac{dX}{\sigma(\Delta)} \right]^{1/p} \leq C \frac{\omega_1(\Delta)}{\sigma(\Delta)}, \quad (S_p)$$

then the Dirichlet problem in $L^{p'}$ is solvable for L_1 .

Proof of (3.1). We must show the *a priori* estimate $\|Nu_1\|_{p'} \leq C \|g\|_{p'}$. Letting u_0 be the solution to the corresponding Dirichlet problem for L_0 , we use integration by parts to write u_1 in terms of u_0 and the following potential F :

$$\begin{aligned} F(X) &= u_1(X) - u_0(X) = \int_{Y \in B_1} G_1(X, Y) L_1(u_1 - u_0)(Y) dY \\ &= \int_{Y \in B_1} \nabla_Y G_1(X, Y) \cdot \varepsilon(Y) \nabla u_0(Y) dY. \end{aligned} \quad (3.1.1)$$

Then $\|Nu_1\|_{p'} \leq \|Nu_0\|_{p'} + \|NF\|_{p'}$, and we seek to bound the second term, as the bound for the first term is known.

Before going further, we note that by Moser's Harnack principle and well-known estimates for the Green's function, we may assume that $L_0 = L_1$ "away from the boundary." We thus need only to bound the potential F over an annulus, say $B_1 \setminus B(0, 3/4)$.

We now break up the potential $F(X)$ into pieces and treat them separately. We note here that this process of breaking up the region B_1 was used in [F-K-P] to bound a similar potential.

To begin, let $Q_0 \in \partial B_1$ and $X \in \Gamma(Q_0)$ be fixed throughout this section. For some pieces of the potential, we will bound F pointwise at X , and for others, we will find a bound for \tilde{N} by bounding F over an average value integral about X . Where the bound for \tilde{N} is obtained, \tilde{N} will be equivalent to N .

We write, for any $Z \in B(X, \delta(X)/4) = B(X)$,

$$\begin{aligned} F(Z) &= \int_{Y \in B(X)} \nabla_Y G_1(Z, Y) \cdot \varepsilon(Y) \nabla u_0(Y) dY \\ &\quad + \int_{Y \in B_1 \setminus B(X)} \nabla_Y G_1(Z, Y) \cdot \varepsilon(Y) \nabla u_0(Y) dY \\ &= F_1(Z) + F_2(Z). \end{aligned} \quad (3.1.2)$$

$F_2(Z)$ will be partitioned further in our later arguments. Our first goal in the course of proving Theorem (3.1) will be to show that

$$\tilde{N}F(Q_0) = \tilde{N}F_1(Q_0) + NF_2(Q_0) \leq CSu_0(Q_0) + C[M(S^{p'}u_0)(Q_0)]^{1/p'}. \quad (3.1.3)$$

We first handle $F_1(Z)$, the part of the potential near the pole of $G_1(X, Y)$; the argument for this part of the estimate follows a very similar argument in [F-K-P].

Define $B(X) = B(X, \delta(X)/4)$, and $2B(X) = B(X, \delta(X)/2)$. For some small fixed ε , let $\tilde{G}_1(Z, Y)$ be the Green's function for L_1 for the ball $B(X, \delta(X)/(2 + \varepsilon))$, and let $K(Z, Y) = G_1(Z, Y) - \tilde{G}_1(Z, Y)$. Then

$$\begin{aligned} F_1(Z) &= \int_{B(X)} \nabla_Y \tilde{G}_1(Z, Y) \cdot \varepsilon(Y) \nabla u_0(Y) dY \\ &\quad + \int_{B(X)} \nabla_Y K(Z, Y) \cdot \varepsilon(Y) \nabla u_0(Y) dY \\ &= \tilde{F}_1 + \tilde{\tilde{F}}_1. \end{aligned}$$

LEMMA 3.3. $\tilde{N}\tilde{F}_1(Q) \leq CSu_0(Q)$.

Proof of (3.3). Note first that by ellipticity and an integration by parts, and since $\tilde{F}_1 = 0$ on $\partial B(X)$ by properties of \tilde{G}_1 , we have

$$\int_{Z \in 2B(X)} |\nabla \tilde{F}_1|^2 dZ \leq C \int_{2B(X)} A_1 \nabla \tilde{F}_1 \cdot \nabla \tilde{F}_1 = -C \int_{2B(X)} \tilde{F}_1 L_1 \tilde{F}_1.$$

Now, letting $\chi_{B(X)}$ denote the characteristic function for the set $B(X)$, we have $L_1 \tilde{F}_1(Z) = -\operatorname{div}(\varepsilon \nabla u_0 \chi_{B(X)}(Z))$ for $Z \in 2B(X)$, since

$$\tilde{F}_1(Z) = - \int_{2B(X)} \tilde{G}_1(Z, Y) \operatorname{div}[\varepsilon(Y) \nabla u_0(Y) \chi_{B(X)}] dY,$$

and since also

$$\int_{2B(X)} \tilde{G}_1(Z, Y) L_1 \tilde{F}_1(Z) = \tilde{F}_1(Z).$$

Thus,

$$\int_{2B(X)} \tilde{F}_1 L_1 \tilde{F}_1 = \int_{2B(X)} \nabla \tilde{F}_1 \cdot \varepsilon(Y) \nabla u_0(Y) \chi_{B(X)} dY.$$

Now boundedness of the coefficients of A_0 and A_1 , together with Cauchy-Schwarz gives

$$\left[\int_{2B(X)} |\nabla \tilde{F}_1|^2 dY \right]^{1/2} \lesssim \left[\int_{B(X)} |\nabla u_0(Y)|^2 dY \right]^{1/2}.$$

Since $\delta(Y) \approx \delta(X)$ for $Y \in B(X)$, we can write

$$\begin{aligned} & \left[\frac{1}{\delta(X)^{n-2}} \int_{2B(X)} |\nabla \tilde{F}_1|^2 dY \right]^{1/2} \\ & \leq \left[\int_{B(X)} |\nabla u_0(Y)|^2 \delta(Y)^{2-n} dY \right]^{1/2} \lesssim Su_0(Q), \end{aligned}$$

where the last inequality follows since $X \in \Gamma(Q)$.

Now apply to \tilde{F}_1 the following form of the Poincaré inequality (cf. [19]): for $B_r = \{|x| < r\}$, $u \in H_0^{1,2}(B_r)$,

$$\int_{B_r} u^2(x) dx \leq Cr^2 \int_{B_r} \nabla_x u^2(x) dx.$$

Thus, we get $\tilde{N}\tilde{F}_1(Q) \lesssim Su_0(Q)$ by taking suprema over $\Gamma(Q)$. ■

LEMMA 3.4. $N\tilde{F}_1 \leq cSu_0(Q)$.

Proof of (3.4). Note that for any Z , $K(Z, Y)$ is a solution to L_1 in $2B(X)$. Thus, by Cacciopoli's inequality (2.9) applied to K ,

$$\begin{aligned} |\tilde{F}_1(Z)| &\leq \frac{C}{\delta(X)} \left[\int_{(3/2)B(X)} |K(Z, Y)|^2 dY \right]^{1/2} \cdot \left[\int_{B(X)} |\nabla u_0(Y)|^2 dY \right]^{1/2} \\ &\lesssim \delta^{n-2}(X) \left[\frac{1}{|(3/2)B(X)|} \int_{(3/2)B(X)} |K(Z, Y)|^2 dY \right]^{1/2} \\ &\quad \cdot \left[\int_{B(X)} |\nabla u_0(Y)|^2 \delta(X)^{2-n} dY \right]^{1/2}. \end{aligned}$$

Now, over $(3/2)B(X)$, we may use Harnack's comparison theorem on the first term above, since K is non-negative. Hence

$$\begin{aligned} |\tilde{F}_1(Z)| &\lesssim \delta^{n-2}(X) \left[\frac{1}{|(3/2)B(X)|} \int_{(3/2)B(X)} K(Z, Y) dY \right] \\ &\quad \times \left[\int_{B(X)} |\nabla u_0(Y)|^2 \delta(X)^{2-n} dY \right]^{1/2} \\ &\lesssim \delta^{n-2}(X) \left[\frac{1}{|(3/2)B(X)|} \int_{(3/2)B(X)} |Z - Y|^{2-n} dY \right] \\ &\quad \times \left[\int_{B(X)} |\nabla u_0(Y)|^2 \delta(X)^{2-n} dY \right]^{1/2} \\ &\lesssim S_\alpha u_0(Q) \lesssim Su_0(Q) \end{aligned}$$

by known estimates for Green's functions, and where α is chosen large enough so that the corresponding cone contains $B(X)$. Note that for the last inequality, we are using the fact that square functions over cones of different apertures are comparable [9]. Thus Lemma 3.4 is complete. ■

With Lemmas 3.3 and 3.4, we have estimated the term F_1 , and we now handle F_2 by breaking up the region $B_1 \setminus B(X)$ further into dyadic ring-type regions, as follows. Let $X^* = X/|X|$, the projection of X onto ∂B_1 , and $\Delta_0 = \Delta(X^*, \delta(X)/2)$. Let $\Omega_0 = B_1 \cap B(X^*, \delta(X)/2)$. Furthermore, for N the smallest integer such that $2 \leq 2^N \delta(X)$, let $j = 1, 2, \dots, N$, and define $\Omega_j = [B_1 \setminus B(X)] \cap B(X^*, 2^{j-1} \delta(X))$ and $R_j = \Omega_j \setminus \Omega_{j-1}$.

Then we can write, for $Z \in 2B(X)$,

$$\begin{aligned} F_2(Z) &= \int_{\Omega_0} \nabla_Y G_1(Z, Y) \cdot \varepsilon(Y) \nabla u_0(Y) dY \\ &\quad + \sum_{j=1}^N \int_{R_j} \nabla_Y G_1(Z, Y) \cdot \varepsilon(Y) \nabla u_0(Y) dY \\ &= F_2^0(Z) + \sum_{j=1}^N F_2^j(Z). \end{aligned} \quad (3.1.4)$$

We now proceed to bound these remaining parts of the potential. From now on, we will find pointwise bounds for F_2 at X , which give us bounds for NF_2 . The essential part of the proof is the bound for $F_2^0(X)$. The additional pieces of the potential will then be estimated by similar methods, and summed up appropriately.

PROPOSITION 3.5. $NF_2^0(Q) \leq C\{M[S^{p'}(u_0)](Q)\}^{1/p'}$.

Proof of (3.5). We begin by using Fubini to write the potential in terms of an integral over the cone $\Gamma_h(X)$ truncated at height $h \approx \text{rad}(\mathcal{A}_0)$.

$$\begin{aligned} |F_2^0(X)| &\leq \int_{x \in \mathcal{A}_0} \int_{Y \in \Gamma_h(x)} |\varepsilon(Y)| |\nabla_Y G_1(X, Y)| |\nabla u_0(Y)| \delta^{1-n}(Y) dY d\sigma(x) \\ &= \int_{x \in \mathcal{A}_0} \sum_{I \subseteq \mathcal{A}_0} l^{1-n}(I) \int_{Y \in \Gamma_h^I(x)} |\varepsilon(Y)| |\nabla_Y G_1(X, Y)| \\ &\quad \times |\nabla u_0(Y)| dY d\sigma(x) \end{aligned}$$

Here the sum takes place over dyadic $I \subseteq \mathcal{A}_0$, and $\Gamma_h^I(x) = I^+ \cap \Gamma_h(x)$, where

$$I^+ = \left\{ Y \in B_1 : \frac{Y}{\delta(Y)} \in I \text{ and } c_n l(I) \leq \delta(Y) \leq 2c_n l(I) \right\}$$

for some constant c_n , where $l(I)$ is the length of I . Then Cauchy-Schwarz gives the bound

$$\begin{aligned} &\int_{x \in \mathcal{A}_0} \sum_{I \subseteq \mathcal{A}_0} \sup_{Y \in \Gamma_h^I(x)} |\varepsilon(Y)| l(I)^{1-n} \left[\int_{Y \in \Gamma_h^I(x)} |\nabla_Y G_1(X, Y)|^2 dY \right]^{1/2} \\ &\quad \cdot \left[\int_{Y \in \Gamma_h^I(x)} |\nabla u_0(Y)|^2 dY \right]^{1/2} d\sigma(x). \end{aligned}$$

Now for each $I \subseteq \mathcal{A}_0$, let Q_I denote the smallest rectangle such that $\Gamma_h^I(x) \subseteq Q_I \subseteq I^+$. Note first that the ratio of the lengths of the sides of these Q_I is fixed. Also, note that $\text{vol}(Q_I) \leq \alpha \text{vol}(\Gamma_h^I(x))$, for some fixed α , independent of I . Both the above facts hold because the aperture of the cone $\Gamma_h(x)$ is fixed. The first fact allows us to use Caccopoli's inequality for integrals over Q_I ; thus $|F_2^0(X)|$ is bounded by a constant multiple of

$$\int_{x \in \mathcal{A}_0} \sum_{I \subseteq \mathcal{A}_0} \sup_{Y \in \Gamma_h^I(x)} |\varepsilon(Y)| \frac{l(I)^{1-n}}{l(I)} \left[\int_{Y \in (3/2)Q_I} |G_1(X, Y)|^2 dY \right]^{1/2} \cdot \left[\int_{Y \in \Gamma_h^I(x)} |\nabla u_0(Y)|^2 dY \right]^{1/2} d\sigma(x). \quad (3.5.1)$$

We now claim the following lemma, which allows us to move the pole of the Green's function from X to the origin.

LEMMA 3.6. *For $Y \in (3/2)Q_I$ and for any $Z \in B(X)$,*

$$\frac{G_1(Z, Y)}{G_1(Y)} \leq \frac{C}{\omega_1(\mathcal{A}_Z)}.$$

Proof of (3.6). Note that $G_1(Z, Y)$ and $G_1(Y)$, as functions of $Y \in (3/2)Q_I$, both satisfy the hypotheses for the Comparison Principle (Theorem 2.4). Thus, for Z_0 such that $\delta(Z_0) = (1/4)\delta(Z)$ and $\delta(Z, Z_0) = (3/4)\delta(Z)$, Theorem 2.5 and standard estimates on the Green's functions (cf. [14]) give

$$\begin{aligned} \frac{G_1(Z, Y)}{G_1(Y)} &\leq C \frac{G_1(Z, Z_0)}{G_1(0, Z_0)} \\ &\lesssim \frac{|Z - Z_0|^{2-n}}{\omega_1^0[\mathcal{A}(Z/|Z|, \delta(Z))]\delta^{2-n}(Z)} \\ &\leq \frac{C}{\omega_1(\mathcal{A}_Z)} \end{aligned}$$

where $\mathcal{A}_Z = \mathcal{A}(Z^*, \delta(Z))$. ■

So Lemma 3.6 together with the observation that $\text{vol}(Q)$ is comparable to $\text{vol}(\Gamma_h^I(X))$ bounds expression (3.5.1) by a constant multiple of

$$\int_{x \in \mathcal{A}_0} \sum_{I \subseteq \mathcal{A}_0} \sup_{Y \in \Gamma_h^I(x)} |\varepsilon(Y)| l(I)^{1-n} \frac{1}{\omega_1(\mathcal{A}_X)} \left[\int_{Y \in \Gamma_h^I(x)} \left| \frac{G_1(Y)^2}{\delta(Y)} \right|^2 dY \right]^{1/2} \cdot \left[\int_{Y \in \Gamma_h^I(x)} |\nabla u_0(Y)|^2 dY \right]^{1/2} d\sigma(x).$$

Hence, using the definition of $a(Y)$ and the doubling of ω_1 ,

$$\begin{aligned}
|F_2^0(X)| &\lesssim \frac{1}{\omega_1(\mathcal{A}_0)} \int_{x \in \mathcal{A}_0} \left[\sum_{I \subseteq \mathcal{A}_0} \int_{Y \in \Gamma_h^I(x)} \frac{a^2(Y)}{\delta^n(Y)} \left| \frac{G_1(Y)}{\delta(Y)} \right|^2 dY \right]^{1/2} \\
&\quad \cdot \left[\sum_{I \subseteq \mathcal{A}_0} \int_{Y \in \Gamma_h^I(x)} |\nabla u_0(Y)|^2 \delta^{2-n}(Y) dY \right]^{1/2} d\sigma(x) \\
&= \frac{1}{\omega_1(\mathcal{A}_0)} \int_{x \in \mathcal{A}_0} \left[\int_{Y \in \Gamma_h(x)} \frac{a^2(Y)}{\delta^n(Y)} \left| \frac{G_1(Y)}{\delta(Y)} \right|^2 dY \right]^{1/2} Su_0(x) d\sigma(x) \\
&\lesssim \frac{1}{\omega_1(\mathcal{A}_0)} |\mathcal{A}_0| \\
&\quad \cdot \left\{ \frac{1}{|\mathcal{A}_0|} \int_{x \in \mathcal{A}_0} \left[\int_{Y \in \Gamma_h(x)} \frac{a^2(Y)}{\delta^n(Y)} \left| \frac{G_1(Y)}{\delta(Y)} \right|^2 dY \right]^{p/2} d\sigma(x) \right\}^{1/p} \\
&\quad \cdot \left\{ \frac{1}{|\mathcal{A}_0|} \int_{x \in \mathcal{A}_0} [Su_0(x)]^{p'} d\sigma(x) \right\}^{1/p'}.
\end{aligned}$$

Now the key step is to apply condition (QS_p) and insert the Hardy–Littlewood maximal function to obtain Proposition 3.5:

$$|F_2^0(X)| \leq C \{ M[(Su_0)^{p'}](Q_0) \}^{1/p'}. \quad \blacksquare$$

The remaining portions of F are:

$$F_2^j(Z) = \int_{R_j} \nabla_Y G_1(Z, Y) \cdot \varepsilon(Y) \nabla u_0(Y) dY$$

for $j = 1, \dots, N$.

For each $j = 1, \dots, N-1$, we further subdivide $R_j = (R_j \cap \Gamma_\gamma(Q_0)) \cup (R_j \setminus \Gamma_\gamma(Q_0))$, where the aperture γ is chosen small enough such that for any $Z \in \Gamma_\gamma(Q_0)$, $Q_0 \in \mathcal{A}(Z^*, \delta(Z)/2)$. Then, we break up F_2^j into

$$\begin{aligned}
F_2^j(Z) &= \int_{R_j \cap \Gamma_\gamma(Q_0)} \nabla_Y G_1(Z, Y) \cdot \varepsilon(Y) \nabla u_0(Y) dY \\
&\quad + \int_{R_j \setminus \Gamma_\gamma(Q_0)} \nabla_Y G_1(Z, Y) \cdot \varepsilon(Y) \nabla u_0(Y) dY \\
&= F_3^j(Z) + F_4^j(Z).
\end{aligned}$$

Consider the $F_3^j(Z)$ first. For notation, let $R_j \cap \Gamma_\gamma(Q_0) = R_j^\Gamma$, $\mathcal{A}_j = \partial\Omega_j \cap \partial B_1$, and let X_j be a point in R_{j+1}^Γ such that $\delta(X_j) \approx 2^{j-1}\delta(X) = r_j$. Then let $Z = X$, our original fixed point in $\Gamma(Q_0)$. The general idea for these regions is to move the pole of the Green's function from X to X_{j+1} ,

and then to 0. Then, bounding the potential F_3^j by an integral over Ω_j , we can argue as for the region Ω_0 . However, moving the pole will introduce a factor of $2^{-j\alpha}$, so we will be able to sum up the F_3^j . We first note that Cauchy–Schwarz plus Cacciopoli gives

$$\begin{aligned} F_3^j(X) &\leq \sum_{I \subseteq A_j} \int_{I^+ \cap R_j^r} |\varepsilon(Y)| |\nabla_Y G_1(X, Y)| |\nabla u_0(Y)| dY \\ &\leq \sum_{I \subseteq A_j} \sup_{I^+ \cap R_j^r} |\varepsilon(W)| \left[\int_{I^+ \cap R_j^r} \frac{|G_1(X, Y)|^2}{r_j^2} dY \right]^{1/2} \\ &\quad \cdot \left[\int_{I^+ \cap R_j^r} |\nabla u_0(Y)|^2 dY \right]^{1/2}. \end{aligned}$$

At this point, we need the following lemma, which allows us to move the pole of the Green's function. The proof will be given later.

LEMMA 3.7. *With all notation as above, there exist constants M , M' and α such that for $Y \in R_j \cap \Gamma_\gamma(Q_0)$, for each j , we have*

$$G_1(X, Y) \leq M 2^{-j\alpha} G_1(X_{j+1}, Y) \leq M' 2^{-j\alpha} \frac{G_1(Y)}{\omega_1(A_j)}$$

Given Lemma 3.7, and noting that $r_j \approx \delta(Y)$ for $Y \in I^+ \cap R_j^r$, arguments just as in Proposition 3.5 show that

$$\begin{aligned} |F_3^j(X)| &\lesssim \frac{2^{-j\alpha}}{\omega_1(A_j)} r_j^{(n-1)/2} \sum_{I \subseteq A_j} \left[\int_{I^+ \cap R_j^r} \left| \frac{G_1(Y)}{\delta(Y)} \right|^2 \frac{a^2(Y)}{\delta(Y)} dY \right]^{1/2} \\ &\quad \cdot \left[\int_{I^+ \cap R_j^r} |\nabla u_0(Y)|^2 \delta^{2-n}(Y) dY \right]^{1/2} \\ &\lesssim \frac{2^{-j\alpha}}{\omega_1(A_j)} |A_j| \left[\frac{1}{|A_j|} \int_{R_j^r} \left| \frac{G_1(Y)}{\delta(Y)} \right|^2 \frac{a^2(Y)}{\delta(Y)} dY \right]^{1/2} Su_0(Q_0). \end{aligned}$$

Now we observe that $R_j^r \subseteq \Gamma_\alpha(Q)$ for all $Q \in A_j$, for a fixed aperture α that does not depend on j . Also, note that $|A_j| \approx \delta^{n-1}(Y)$ for $Y \in R_j^r$. Thus, the above is bounded by

$$\begin{aligned} &\frac{2^{-j\alpha}}{\omega_1(A_j)} |A_j| \inf_{Q \in A_j} \left[\int_{\Gamma_\alpha(Q)} \left| \frac{G_1(Y)}{\delta(Y)} \right|^2 \frac{a^2(Y)}{\delta^n(Y)} dY \right]^{1/2} Su_0(Q_0) \\ &\lesssim \frac{2^{-j\alpha}}{\omega_1(A_j)} |A_j| \frac{1}{|A_j|} \int_{Q \in A_j} \left[\int_{\Gamma_\alpha(Q)} \left| \frac{G_1(Y)}{\delta(Y)} \right|^2 \frac{a^2(Y)}{\delta^n(Y)} dY \right]^{1/2} d\sigma(Q) Su_0(Q_0), \end{aligned}$$

which by Hölder's inequality and the key condition (QS_p) is bounded by $C2^{-j\alpha}Su_0(Q_0)$, where C is independent of j and X . Hence the F_3^j can be summed over j to get

$$\sum_j F_3^j(X) \leq CSu_0(Q_0),$$

where C is independent of the point X .

To complete the argument for the estimate of the F_3^j , it remains to prove Lemma 3.7.

Proof of (3.7). First note that $G_1(X, Y)$ is a positive L_1 -harmonic function in Y away from X , and that $G_1(X_j, Y)$ is a positive L_1 -harmonic function in Y away from X_j ; furthermore, both these functions vanish for $Y \in \partial B_1$.

Now from [3] together with well-known arguments (such as in [18]), a solution u vanishing on a surface ball $\Delta(Q, r) \subseteq \partial B_1$ has the following Hölder continuity with decay at the boundary for $Z \in B_1 \cap B(Q, r)$:

$$u(Z) \leq M \left(\frac{|Z - Q|}{r} \right)^\alpha \sup\{u(Y) : Y \in B(Q, r) \cap B_1\}.$$

Thus, taking Z to be X , our original fixed point, and taking $B(Q, r) = B(X^*, 2^{j-1}\delta(X))$, we obtain

$$\begin{aligned} G_1(X_j, X) &\leq M \left(\frac{|X - X^*|}{2^{j-1}\delta(X)} \right)^\alpha \sup\{G_1(X_j, Y) : Y \in B(X^*, 2^{j-1}\delta(X)) \cap B_1\} \\ &\lesssim M 2^{-j\alpha} CG_1(X_j, X_{j-1}). \end{aligned}$$

Now by Harnack, in fact $G_1(X_j, X_{j-1}) \approx G_1(X_j, X_{j+1})$.

Furthermore, by the comparison principle (Theorem 2.4), we have

$$\frac{G_1(X, Y)}{G_1(X_{j+1}, Y)} \approx \frac{G_1(X, X_j)}{G_1(X_{j+1}, X_j)} \quad \text{for } Y \in \Gamma_\gamma(Q_0) \cap R_j.$$

Now, the comparison principle (Theorem 2.4) shows that for $Y \in \Gamma_\gamma(Q_0) \cap R_j$, we have $G_1(X, Y) \approx (G_1(X, X_j)/G_1(X_{j+1}, X_j)) \cdot G(X_{j+1}, Y)$, which by the above estimate plus Harnack is bounded by a constant times $2^{-j\alpha}G(X_{j+1}, Y)$.

Finally, for $Y \in \Gamma_\gamma(Q_0) \cap R_j$, we have

$$G_1(X_{j+1}, Y) \leq C \frac{G_1(Y)}{\omega_1(\Delta_j)},$$

by Lemma 3.6 applied to $Y \in \Gamma_\gamma(Q_0) \cap R_j$ and $Z = X_{j+1}$, together with the doubling of ω_1 . Thus we obtain Lemma 3.7. ■

At this point, we have estimated all pieces of the potential except the $F_4^j(X) = \int_{R_j \setminus \Gamma_\gamma(Q_0)} \nabla_Y G_1(X, Y) \cdot \varepsilon(Y) \nabla u_0(Y) dY$, $j = 1, \dots, N-1$. To handle these pieces, we segment $R_j \setminus \Gamma_\gamma(Q_0)$ into K comparable sections each having length $\approx r_j = 2^{j-1} \delta(X)$. That is, let S_r^k be the k -dimensional sphere of radius r . Then, for $n=3$, $S_{r_j}^1$ has surface measure $2\pi r_j$, so we can take K to be the smallest integer larger than 2π . In general, we take $K \approx \sigma_{n-2}$ where σ_{n-2} is the surface measure of the $(n-2)$ -dimensional unit sphere. Thus K depends only on n .

The idea is to note that $R_j \subset \Omega_j$ and break $\Omega_j \setminus \Gamma_\gamma(Q_0)$ into $J = 1, \dots, K$ regions over Δ_j^J which are each essentially Carleson regions over surface balls of radius $\approx \frac{1}{2} r_j$. The potential over each of these Carleson regions is estimated as the previous pieces were. We let $\Omega_j \setminus \Gamma_\gamma(Q_0) = \bigcup_{J=1}^K \tilde{T}(\Delta_j^J)$, where $X \notin \tilde{T}(\Delta_j^J)$. Then for each j and each J , we have

$$\begin{aligned} & \int_{\tilde{T}(\Delta_j^J)} \nabla_Y G_1(X, Y) \cdot \varepsilon(Y) \nabla u_0(Y) dY \\ & \leq \int_{x \in \Delta_j^J} \int_{Y \in \Gamma_h(x)} |\varepsilon(Y)| |\nabla_Y G_1(X, Y)| |\nabla u_0(Y)| \delta^{1-n}(Y) dY d\sigma(x) \\ & \leq \int_{x \in \Delta_j^J} \sum_{I \subseteq \Delta_j^J} \sup_{Y \in \Gamma_h^I(x)} |\varepsilon(Y)| l(I)^{1-n} \left[\int_{Y \in \Gamma_h^I(x)} |\nabla_Y G_1(X, Y)|^2 dY \right]^{1/2} \\ & \quad \cdot \left[\int_{Y \in \Gamma_h^I(x)} |\nabla u_0(Y)|^2 dY \right]^{1/2} d\sigma(x). \end{aligned}$$

Now, by applying Cacciopoli, moving the pole of the Green's function by Lemma 3.7, and using the definition of $a(Y)$ as before, we obtain the bound

$$\begin{aligned} & \int_{x \in \Delta_j^J} \sum_{I \subseteq \Delta_j^J} \sup_{Y \in \Gamma_h^I(x)} |\varepsilon(Y)| l(I)^{1-n} \frac{2^{-j\alpha}}{\omega_1(\Delta_j^J)} \left[\int_{Y \in \Gamma_h^I(x)} \left| \frac{G_1(Y)}{\delta(Y)} \right|^2 dY \right]^{1/2} \\ & \quad \cdot \left[\int_{Y \in \Gamma_h^I(x)} |\nabla u_0(Y)|^2 dY \right]^{1/2} d\sigma(x) \\ & \leq C \frac{2^{-j\alpha}}{\omega_1(\Delta_j^J)} \int_{x \in \Delta_j^J} \sum_{I \subseteq \Delta_j^J} \left[\int_{Y \in \Gamma_h^I(x)} \left| \frac{G_1(Y)}{\delta(Y)} \right|^2 \frac{a^2(Y)}{\delta^n(Y)} dY \right]^{1/2} \\ & \quad \cdot \left[\int_{Y \in \Gamma_h^I(x)} |\nabla u_0(Y)|^2 \delta^{2-n}(Y) dY \right]^{1/2} d\sigma(x) \end{aligned}$$

$$\begin{aligned}
&\leq C \frac{2^{-j\alpha}}{\omega_1(A_j^J)} \int_{x \in A_j^J} \left[\int_{Y \in \Gamma_h(x)} \left| \frac{G_1(Y)}{\delta(Y)} \right|^2 \frac{a^2(Y)}{\delta^n(Y)} dY \right]^{1/2} Su_0(x) d\sigma(x) \\
&\leq C \frac{2^{-j\alpha}}{\omega_1(A_j^J)} |A_j^J| \left\{ \frac{1}{|A_j^J|} \int_{A_j^J} \left[\int_{\Gamma_h(x)} \left| \frac{G_1(Y)}{\delta(Y)} \right|^2 \frac{a^2(Y)}{\delta^n(Y)} dY \right]^{p/2} dx \right\}^{1/p} \\
&\quad \cdot \left\{ \frac{1}{|A_j^J|} \int_{A_j^J} [Su_0(x)]^{p'} d\sigma(x) \right\}^{1/p'}.
\end{aligned}$$

Now, again by the condition (QS_p) plus doubling of surface measure and insertion of the Hardy–Littlewood maximal function, we have the bound

$$C 2^{-j\alpha} \{M[(Su_0)^{p'}](Q_0)\}.$$

Hence, when we sum over $J = 1, \dots, K$ and $j = 1, \dots, N-1$, we get

$$\sum_{j=1}^{N-1} F_4^j(X) \leq C \{M[(Su_0)^{p'}](Q_0)\}^{1/p'}$$

where C is independent of X .

Finally, we bound the last piece, $F_2^N(X)$, the potential over $R_N = \Omega_N \setminus \Omega_{N-1}$. Note that if we let $R = B_1 \setminus B(Q_0, \frac{1}{2})$, then $R_n \subseteq R_1$ and it is enough to bound the expression

$$\int_{R \setminus B(0, 1/2)} |\nabla_Y G_1(X, Y) \cdot \varepsilon(Y) u_0(Y)| dY.$$

We require a lemma which will enable us to use the condition (QS_p) without moving the pole of the Green's function.

LEMMA 3.8. *For $Y \in R \setminus B(0, \frac{1}{2})$, if (QS_p) holds, then for $\Delta \subseteq \partial B_1 \cap \partial R$, there is a constant C such that the following condition also holds:*

$$\left\{ \int_{Q \in \Delta} \left[\int_{Y \in \Gamma(Q)} \frac{a^2(Y)}{\delta(Y)} \left(\frac{G_1(X, Y)}{\delta(Y)} \right)^2 dY \right]^{p/2} \frac{d\sigma(Q)}{\sigma(\Delta)} \right\}^{1/p} \leq C \frac{\omega_1^X(\Delta)}{\sigma(\Delta)}. \quad (3.9.1)$$

Proof of (3.8). First, note that for $Y_0 \in T(\Delta)$ such that $\delta(Y_0) \approx \text{rad}(\Delta)$, the comparison theorem gives us

$$\frac{G_1(X, Y)}{G_1(Y)} \leq \frac{G_1(X, Y_0)}{G_1(Y_0)} \approx \frac{\omega_1^X(\Delta)}{\omega_1(\Delta)}.$$

Thus,

$$\left\{ \int_{Q \in \mathcal{A}} \left[\int_{Y \in \Gamma(Q)} \frac{a^2(Y)}{\delta(Y)} \left(\frac{G_1(X, Y)}{\delta(Y)} \right)^2 \left(\frac{G_1(Y)}{G_1(Y)} \right)^2 dY \right]^{p/2} \frac{d\sigma(Q)}{\sigma(\mathcal{A})} \right\}^{1/p}$$

$$\leq \left\{ \int_{Q \in \mathcal{A}} \left[\int_{Y \in \Gamma(Q)} \left(\frac{\omega_1^X(\mathcal{A})}{\omega_1(\mathcal{A})} \right)^2 \frac{a^2(Y)}{\delta(Y)} \left(\frac{G_1(Y)}{\delta(Y)} \right)^2 dY \right]^{p/2} \frac{d\sigma(Q)}{\sigma(\mathcal{A})} \right\}^{1/p},$$

which by (QS_p) is bounded by $C(\omega_1^X(\mathcal{A})/\omega_1(\mathcal{A}))$. ■

We now use Lemma 3.8 to bound F_2^N , proceeding in a manner similar to the argument for F_2^0 (the part over the original Carleson region Ω_0), except that we will not move the pole of $G_1(X, Y)$. The region $R \setminus B(0, \frac{1}{2})$ can be broken up into $k = 1, \dots, K$ Carleson type regions $T(\mathcal{A}^k)$ of length $\approx \frac{1}{2}$, where K depends only on the dimension n . Then for each k , now familiar methods used repeatedly above give

$$\int_{T(\mathcal{A}^k)} |\nabla_Y G_1(X, Y) \cdot \varepsilon(Y) \nabla u_0(Y)| dY$$

$$\lesssim |\mathcal{A}^k| \left\{ \frac{1}{|\mathcal{A}^k|} \int_{\mathcal{A}^k} \left[\int_{\Gamma_h(x)} \left| \frac{G_1(X, Y)}{\delta(Y)} \right|^2 \frac{a^2(Y)}{\delta^n(Y)} dY \right]^{p/2} d\sigma(x) \right\}^{1/p}$$

$$\cdot \left\{ \frac{1}{|\mathcal{A}^k|} \int_{\mathcal{A}^k} [Su_0(x)]^{p'} d\sigma(x) \right\}^{1/p'},$$

which by Lemma 3.8 is

$$\lesssim \omega_1^X(\mathcal{A}^k) \cdot \left\{ \frac{1}{|\mathcal{A}^k|} \int_{\mathcal{A}^k} [Su_0(x)]^{p'} d\sigma(x) \right\}^{1/p'}.$$

Now $\omega_1^X(\mathcal{A}^k) \leq \omega_1^X(\partial B_1) \leq 1$, so we get the bound

$$C \left\{ \frac{1}{|\mathcal{A}^k|} \int_{x \in \mathcal{A}^k} [Su_0(x)]^{p'} d\sigma(x) \right\}^{1/p'}.$$

By doubling, the above is bounded by bounded by $C[M(S^{p'}(u_0))(Q_0)]^{1/p'}$, and we are done.

Altogether, we have shown the estimate

$$\tilde{N}F(Q_0) = \tilde{N}F_1(Q_0) + NF_2(Q_0)$$

$$\leq CSu_0(Q_0) + C[M(S^{p'}u_0)(Q_0)]^{1/p'},$$

which is the estimate (3.1.3) we wished to obtain as our first step towards proving Theorem 3.1. To complete the proof, we show that the estimate

(3.1.3) in fact gives the estimate $\|Nu_1\|_{L^{p'}} \leq C \|g\|_{L^{p'}}$, where g is continuous boundary data in $L^{p'}$. Once we have this *a priori* estimate, standard arguments, which will not be given here, directly give existence of non-tangential limits. Examples of such arguments appear for instance in [24] and [18].

We have written

$$\begin{aligned} u_1(Z) &= u_0(Z) + \int_D \nabla_Y G_1(Z, Y) \cdot \varepsilon \nabla u_0(Y) dY \\ &= u_0(Z) + F(Z), \end{aligned}$$

and have shown

$$Nu_1(Q) \lesssim Su_0(Q) + [M(S^{p'}u_0)(Q)]^{1/p'}.$$

We will be using the following terminology: by a “strong type (p, p) ” estimate for an operator T on a function f on ∂B_1 , we will mean that the estimate $\|Tf\|_{L^p} \leq C \|f\|_{L^p}$ is satisfied. By a “weak type (p, p) ” estimate we will mean that the following is satisfied: $|\{Tf > \lambda\}| \leq (C/\lambda^p) \|f\|_{L^p}^p$.

Claim 3.9. Given the above inequality $Nu_1(Q) \lesssim Su_0(Q) + [M(S^{p'}u_0)(Q)]^{1/p'}$, $Nu_1 = Nu_0 + \tilde{N}F$ satisfies a weak type (p', p') estimate.

Proof of (3.9). Note that $\{Nu_1 > \lambda\} = \{\tilde{N}u_1 > \lambda\} \subseteq \{Nu_0 > \lambda/2\} \cup \{\tilde{N}F > \lambda/2\}$. A weak type (p', p') estimate holds for Nu_0 , as u_0 is a solution. For $\tilde{N}F$, we have shown by (3.1.3) that

$$(\tilde{N}F)^{p'} \leq CM_\sigma[S^{p'}(u_0)]. \quad (3.9.1)$$

Now, the maximal function $M = M_\sigma$ satisfies a weak-type $(1, 1)$ estimate (cf. [23]), and furthermore u_0 is a solution. Thus we obtain

$$|\{\tilde{N}F > \lambda\}| \lesssim \frac{1}{\lambda^{p'}} \int_{\partial B} S^{p'}(u_0) d\sigma \lesssim \frac{1}{\lambda^{p'}} \int_{\partial B} g^{p'} d\sigma,$$

where the last estimate follows from Theorem 2.10. ■

The key step now is to note that the weak-type estimate of Claim 3.9 leads to a weak-type (p', p') estimate (with respect to surface measure) for M_{ω_1} . That is,

$$|\{M_{\omega_1} g > \lambda\}| \leq \frac{C}{\lambda^{p'}} \int_{\partial B_1} g^{p'} d\sigma, \quad (3.10)$$

where

$$M_{\omega_1} g(Q) = \sup_{A \ni Q} \frac{1}{\omega_1(A)} \int_A g \, d\omega_1.$$

The above weak-type estimate follows immediately from Claim 3.9 together with the point-wise estimate that $M_{\omega_1} g(Q_0) \leq C N u_1(Q_0)$ for any $Q_0 \in \partial B_1$, a fact that is well known from the methods of Caffarelli *et al.*, as shown in [3], based on ideas first used by Hunt and Wheeden [15, 16].

To complete our proof, we apply results of Muckenhoupt which show that the weak type estimate on M_{ω_1} gives the strong type estimate: $\|M_{\omega_1} u_1\|_{L^{p'}(d\sigma)} \leq \|g\|_{L^{p'}(d\sigma)}$. We state below a special case of Muckenhoupt's results. These results hold both for weights on \mathbb{R}^n and on ∂B_1 , where B_1 is the unit ball in \mathbb{R}^n , as the same methods are valid in both settings.

THEOREM 3.11 [22]. *Say $1 < q < \infty$, and $0 < \lambda < \infty$. Let m be a Borel measure on $J \subseteq \partial B_1$ such that m is 0 on sets consisting of single points. Let $U(x)$ be a weight (i.e., a nonnegative function) on J . Given a function $f(x)$ defined on J , vanishing on $\partial B_1 \setminus J$, the following are equivalent:*

1. *There exists a constant C_1 , independent of f , such that*

$$\int_{\{M_m f(x) > \lambda\}} U(x) \, dm(x) \leq \frac{C_1}{\lambda^q} \int_J |f(x)|^q U(x) \, dm(x).$$

2. *There exists a constant C , independent of f , such that*

$$\int_J [M_m f(x)]^q U(x) \, dm(x) \leq C \int_J |f(x)|^q U(x) \, dm(x).$$

We apply this theorem with $m = \omega_1$ and $U(x) = k_1^{-1}(x)$, where k_1 is defined by $d\omega_1 = k_1 \, d\sigma$. Then our weak type estimate (3.10) is in fact equivalent to $\|M_{\omega_1} g(x)\|_{L^{p'}(d\sigma)} \leq C \|g\|_{L^{p'}(d\sigma)}$. Furthermore, by [3] we know that $\|M_{\omega_1} g(x)\|_{L^{p'}(d\sigma)} \gtrsim \|N u_1\|_{L^{p'}(d\sigma)}$, which completes the proof of Theorem 3.1. ■

4. AN ALTERNATIVE CONDITION FOR THE PRESERVATION OF SOLVABILITY

We now prove Theorem 3.2. The proof is similar to the proof of Theorem 3.1, as we will again write $u_1(X) = u_0(X) + F(X)$ and demonstrate an L^p bound for NF by breaking $F(X)$ into pieces. However, the estimation of these pieces is complicated by the need for “stopping time” arguments. In order to apply such arguments, we will first need to show that

$(a^2(X)/\delta(X)) dX$ is a Carleson measure. We begin with the following lemma, which is a crucial step in proving the Carleson measure property.

LEMMA 4.1. *For any $p > 1$, if the condition (S_p) holds, then (S_1) holds also; that is, the condition holds when $p = 1$.*

Proof of (4.1). We will use the following notation: for a surface ball Δ , given a dyadic subcube $I \subseteq \Delta$, let

$$I^+ = \left\{ Y \in B_1 : \frac{Y}{\delta(Y)} \in I \text{ and } c_n l(I) \leq \delta(Y) \leq 2c_n l(I) \right\}$$

for some constant c_n , where $l(I)$ is the length of I . Noting that by (2.5),

$$\frac{G_1(Y)}{\delta(Y)} \approx \frac{\omega_1(I)}{\delta^{n-1}(X)} \approx \frac{\omega_1(I)}{\sigma(I)}$$

for $X \in I^+$, (S_p) is then equivalent to

$$\frac{1}{\sigma(\Delta)} \sum_{I \subseteq \Delta} A_I \left[\frac{\omega_1(I)}{\sigma(I)} \right]^p \leq K_p \left[\frac{\omega_1(\Delta)}{\sigma(\Delta)} \right]^p$$

where K_p is some constant and

$$A_I = \int_{I^+} \frac{a^2(X)}{\delta(X)} dX.$$

Let

$$\mathcal{S} = \frac{1}{\sigma(\Delta)} \sum_{I \subseteq \Delta} A_I \left[\frac{\omega_1(I)}{\sigma(I)} \right].$$

We wish to estimate \mathcal{S} in terms of $\omega_1(\Delta)/\sigma(\Delta)$. We begin by subdividing Δ in stages and classifying all the dyadic intervals $I \subseteq \Delta$ as follows. Fix an α , $0 < \alpha < 1$. If I satisfies

$$\frac{\omega(I)}{|I|} < \alpha \frac{\omega(\Delta)}{|\Delta|},$$

then we say $I \in \mathcal{B}_\Delta$, and stop subdividing. Otherwise, put $I \in \mathcal{G}_\Delta$ and subdivide I again and continue to classify as above. Then,

$$\mathcal{S} = \frac{1}{\sigma(\Delta)} \left[\sum_{I \in \mathcal{B}_\Delta} A_I \frac{\omega_1(I)}{|I|} + \sum_{I \in \mathcal{G}_\Delta} A_I \frac{\omega_1(I)}{|I|} + \sum_{I \in \mathcal{B}_\Delta} \sum_{I_j \subseteq I} A_{I_j} \frac{\omega_1(I_j)}{|I_j|} \right].$$

The last term sums over all the remaining intervals that have not been classified in this first stage.

Now consider the $I_j \subseteq I$ where $I \in \mathcal{B}_A$. If I_j is such that $\omega_1(I_j)/|I_j| < \alpha(\omega_1(I)/|I|)$, put it in \mathcal{B}_I^2 and stop. Otherwise, put I_j in \mathcal{G}_I^2 and continue subdividing. After this second stage, we have:

$$\begin{aligned} \mathcal{S} = & \frac{1}{\sigma(\Delta)} \left\{ \sum_{I \in \mathcal{B}_A} A_I \frac{\omega_1(I)}{|I|} + \sum_{I \in \mathcal{G}_A} A_I \frac{\omega_1(I)}{|I|} \right. \\ & \left. + \sum_{I \in \mathcal{B}_A} \left[\sum_{I_j \in \mathcal{G}_I^2} A_{I_j} \frac{\omega_1(I_j)}{|I_j|} + \sum_{I_j \in \mathcal{B}_I^2} A_{I_j} \frac{\omega_1(I_j)}{|I_j|} + \sum_{J \in \mathcal{B}_I^2} \sum_{J_j \subseteq J} A_{J_j} \frac{\omega_1(J_j)}{|J_j|} \right] \right\}. \end{aligned}$$

We continue this process through the n th stage.

Consider the sums over the intervals in the \mathcal{B} classes first. The term from the first stage is:

$$\frac{1}{|\Delta|} \sum_{I \in \mathcal{B}_A} A_I \frac{\omega_1(I)}{|I|}.$$

Now $A_I \lesssim \|a^2(X)\|_\infty |I| = K_a |I|$, so

$$\frac{1}{|\Delta|} \sum_{I \in \mathcal{B}_A} A_I \frac{\omega_1(I)}{|I|} \leq \frac{1}{|\Delta|} K_a \sum_{I \in \mathcal{B}_A} |I| \frac{\omega_1(I)}{|I|}$$

which by the definition of \mathcal{B}_A is

$$\begin{aligned} & \leq \frac{1}{|\Delta|} K_a \sum_{I \in \mathcal{B}_A} \alpha \frac{|I|}{|\Delta|} \omega_1(\Delta) \\ & \leq K\alpha \frac{\omega_1(\Delta)}{|\Delta|}. \end{aligned}$$

The last inequality holds since the $I \in \mathcal{B}_A$ are all disjoint and are contained in Δ .

For the second stage \mathcal{B} term we again use the definition of \mathcal{B}_I to obtain:

$$\begin{aligned} \frac{1}{|\Delta|} \sum_{I \in \mathcal{B}_A} \sum_{I_j \in \mathcal{B}_I^2} A_{I_j} \frac{\omega_1(I_j)}{|I_j|} & \leq \frac{1}{|\Delta|} K_a \sum_{I \in \mathcal{B}_A} \sum_{I_j \in \mathcal{B}_I^2} \omega_1(I_j) \\ & \leq \frac{1}{|\Delta|} K_a \sum_{I \in \mathcal{B}_A} \sum_{I_j \in \mathcal{B}_I^2} \alpha |I_j| \frac{\omega_1(I)}{|I|} \\ & \leq \frac{1}{|\Delta|} K_a \alpha \sum_{I \in \mathcal{B}_A} \omega_1(I) \\ & \leq K_a \alpha^2 \frac{\omega_1(\Delta)}{|\Delta|}. \end{aligned}$$

The last inequality comes from the first stage result.

Similarly, for the n th stage of \mathcal{B} terms, we get

$$\frac{1}{|\mathcal{A}|} \sum_{I \in \mathcal{B}_A} \sum_{J \in \mathcal{B}_I^2} \cdots \sum_{L_j \in \mathcal{B}_L^n} A_{L_j} \frac{\omega_1(L_j)}{|L_j|} \leq K_a \alpha^n \frac{\omega(\mathcal{A})}{|\mathcal{A}|}.$$

Hence altogether, the contribution from all the \mathcal{B} terms is bounded by

$$K_a \frac{\omega_1(\mathcal{A})}{|\mathcal{A}|} \left(\sum_{n=1}^{\infty} \alpha^n \right) = K_a \left(\frac{\alpha}{1-\alpha} \right) \frac{\omega_1(\mathcal{A})}{|\mathcal{A}|}.$$

For the \mathcal{G} intervals, we have the following: for the first stage, for $I \in \mathcal{G}_A$, the definition of \mathcal{G}_A gives

$$\left(\frac{\omega_1(\mathcal{A})}{|\mathcal{A}|} \right)^{p-1} \leq \left(\frac{1}{\alpha} \right)^{p-1} \left(\frac{\omega_1(I)}{|I|} \right)^{p-1}.$$

Then, for the first stage term, by the definition of \mathcal{G} ,

$$\frac{1}{|\mathcal{A}|} \sum_{I \in \mathcal{G}_A} A_I \frac{\omega_1(I)}{|I|} \leq \left(\frac{|\mathcal{A}|}{\omega_1(\mathcal{A})} \right)^{p-1} \left(\frac{1}{\alpha} \right)^{p-1} \left[\frac{1}{|\mathcal{A}|} \sum_{I \in \mathcal{G}_A} A_I \left(\frac{\omega_1(I)}{|I|} \right)^p \right].$$

Now we apply the condition (S_p) to bound the above by

$$K_p \left(\frac{1}{\alpha} \right)^{p-1} \frac{\omega_1(\mathcal{A})}{|\mathcal{A}|}.$$

For the \mathcal{G} class terms at the second stage we have

$$\begin{aligned} & \frac{1}{|\mathcal{A}|} \sum_{I \in \mathcal{B}_A} \sum_{I_j \in \mathcal{G}_I^2} A_{I_j} \frac{\omega_1(I_j)}{|I_j|} \\ & \leq \left(\frac{1}{\alpha} \right)^{p-1} \frac{1}{|\mathcal{A}|} \sum_{I \in \mathcal{B}_A} \left(\frac{|I|}{\omega_1(I)} \right)^{p-1} \sum_{I_j \in \mathcal{G}_I^2} A_{I_j} \left(\frac{\omega_1(I_j)}{|I_j|} \right)^p, \end{aligned}$$

which by the condition (S_p) is bounded by

$$\left(\frac{1}{\alpha} \right)^{p-1} K_p \frac{1}{|\mathcal{A}|} \sum_{I \in \mathcal{B}_A} \omega_1(I) \alpha \leq \left(\frac{1}{\alpha} \right)^{p-1} K_p \alpha \frac{\omega_1(\mathcal{A})}{|\mathcal{A}|}.$$

Similarly, at the n th stage, we have

$$\frac{1}{|\mathcal{A}|} \sum_{I \in \mathcal{B}_A} \sum_{J \in \mathcal{B}_I^2} \cdots \sum_{L_j \in \mathcal{G}_L^n} A_{L_j} \frac{\omega_1(L_j)}{|L_j|} \leq \left(\frac{1}{\alpha} \right)^{p-1} K_p \alpha^{n-1} \frac{\omega_1(\mathcal{A})}{|\mathcal{A}|}.$$

Thus,

$$\mathcal{S} \leq \left[K_a \left(\frac{\alpha}{1-\alpha} \right) + \left(\frac{1}{\alpha} \right)^{p-1} K_p \left(\frac{1}{1-\alpha} \right) \right] \cdot \frac{\omega_1(\mathcal{A})}{|\mathcal{A}|},$$

so that (S_1) holds, and we have shown Lemma 4.1. ■

Now with Lemma 4.1 completed, we show:

LEMMA 4.2. $(S_p) \Rightarrow (S_q)$ for $1 \leq q < p$.

Proof of (4.2). Since (S_p) holds, (S_1) holds by Lemma 4.1. Now

$$\begin{aligned} & \frac{1}{|\mathcal{A}|} \int_{T(\mathcal{A})} \frac{a^2(X)}{\delta(X)} \left| \frac{G_1(X)}{\delta(X)} \right|^q dX \\ &= \frac{1}{|\mathcal{A}|} \int_{T(\mathcal{A})} \frac{a^2(X)}{\delta(X)} \left| \frac{G_1}{\delta(X)} \right|^{(p-q)/(p-1)} \left| \frac{G_1}{\delta(X)} \right|^{(pq-p)/(p-1)} dX \end{aligned}$$

which by Hölder's inequality with exponent $(p-1)/(p-q)$ is

$$\begin{aligned} & \leq \left[\frac{1}{|\mathcal{A}|} \int_{T(\mathcal{A})} \frac{a^2(X)}{\delta(X)} \left| \frac{G_1}{\delta(X)} \right| dX \right]^{(p-q)/(p-1)} \\ & \quad \cdot \left[\frac{1}{|\mathcal{A}|} \int_{T(\mathcal{A})} \frac{a^2(X)}{\delta(X)} \left| \frac{G_1}{\delta(X)} \right|^p dX \right]^{(q-1)/(p-1)}. \end{aligned}$$

Finally, (S_p) and (S_1) give the bound $K_1^{(p-q)/(p-1)} K_p^{(q-1)/(p-1)} \times (\omega_1(\mathcal{A})/|\mathcal{A}|)^q$. ■

We are now ready to show that $(a^2(X)/\delta(X)) dX$ is Carleson. This Carleson measure property will allow us to prove Theorem 3.2 by using the condition (S_p) to bound the potential F in a manner similar to the proof of Theorem 3.1.

PROPOSITION 4.3. *The condition (S_p) implies that $(a^2(X)/\delta(X)) dX$ is a Carleson measure.*

Proof of (4.3). First, by Lemma 4.1, (S_p) implies (S_1) . Then we claim that (S_1) implies that, for any $r > 0$,

$$\int_{Q \in \mathcal{A}_r} S_r^2(a) \frac{d\omega_1(Q)}{\omega_1(\mathcal{A}_r)} \leq C, \quad (4.3.1)$$

where $S_r^2(a) = \int_{\Gamma_r(Q)} (a^2(X)/\delta^n(X)) dX$. For, by (S_1) and Theorem 2.5,

$$C \gtrsim \int_{X \in \Gamma_r(Q)} \frac{a^2(X)}{\delta(X)} \frac{\omega_1(\Delta_X)}{\delta^{n-1}(X)} \frac{dX}{\omega_1(\Delta_r)},$$

which by changing the order of integration is comparable to

$$\int_{Q \in \Delta_r} \int_{X \in \Gamma_r(Q)} \frac{a^2(X)}{\delta^n(X)} dX \frac{d\omega_1(Q)}{\omega_1(\Delta_r)}.$$

We must show that the inequality (4.3.1) in fact implies the same inequality with σ replacing ω_1 . Then, changing the order of integration gives precisely the statement that $(a^2(X)/\delta(X)) dX$ is a Carleson measure. By a standard argument (as used in [12], Theorem 2.18), it is known that the desired Carleson inequality will hold if $d\sigma \in A_\infty(d\omega_1)$. Thus it remains to show that $d\sigma \in A_\infty(d\omega_1)$. This fact can be obtained directly from the following result from [12] (Theorem 2.20 of that reference).

THEOREM 4.4 [12]. *Let L_0 and L_1 be two elliptic operators, with associated harmonic measures μ_0 and μ_1 . Let $a(X)$ be the disagreement function for L_0 and L_1 , as usual. Let μ be a doubling measure on ∂B_1 , and suppose $\mu_0 \in A_\infty(d\mu)$. If*

$$\sup_{A \subseteq \partial B_1} \int_A \int_{\Gamma_r(Q)} \frac{a^2(X)}{\delta^n(X)} dX \frac{d\mu(Q)}{\mu(A)} \leq C,$$

then $\mu_1 \in A_\infty(d\mu)$.

We use this theorem with $d\mu = d\omega_1$ and $d\mu_0 = d\omega_1$ and $d\mu_1 = d\omega_0$. Then the theorem says that since we have, by (4.3.2),

$$\sup_{A \subseteq \partial B_1} \int_A \int_{\Gamma_r(Q)} \frac{a^2(X)}{\delta^n(X)} \frac{d\omega_1}{\omega_1(A)} \leq C$$

and also,

$$d\omega_1 \in A_\infty(d\omega_1),$$

we obtain $d\omega_0 \in A_\infty(d\omega_1)$. Finally, the transitivity of A_∞ (cf. [4]) and the fact that $d\omega_0 \in A_\infty(d\sigma)$ gives $d\sigma \in A_\infty(d\omega_1)$, and Proposition 4.3 is proved. ■

Proof of (3.2). As in the proof of Theorem 3.1, we proceed by writing u_1 in terms of u_0 and the potential F . To demonstrate the estimate $\|NF\|_{p'} \leq C \|u_0\|_{p'}$ near the boundary of the domain B_1 , we break up the

potential F into the same pieces as in the proof of Theorem 3.1. Let $B(X) = B(X, \delta(X)/4)$ and write, for any $Z \in B(X)$,

$$\begin{aligned} F(Z) &= \int_{Y \in B(X, \delta(X)/4) = B(X)} \nabla_Y G_1(Z, Y) \cdot \varepsilon(Y) \nabla u_0(Y) dY \\ &\quad + \int_{Y \in B_1 \setminus B(X)} \nabla_Y G_1(Z, Y) \cdot \varepsilon(Y) \nabla u_0(Y) dY \\ &= F_1(Z) + F_2(Z). \end{aligned}$$

Note that to obtain a bound for F_1 , the part of the potential near the pole of $G_1(X, Y)$, an argument identical to the proof of Lemmas 3.3 and 3.4 goes through, and we obtain the pointwise bound $\tilde{N}F_1(Q) \leq C S u_0(Q)$.

To handle F_2 we break up the region $B_1 \setminus B(X)$ further, as before, and write, for $Z \in 2B(X)$,

$$F_2(Z) = F_2^0(Z) + \sum_{j=1}^N F_2^j(Z),$$

where all notation is as in the proof of Theorem 3.1. From now on, we will find pointwise bounds for F_2 at X , which will give us bounds for NF_2 .

In order to bound $F_2^0(X)$, we will use a stopping time argument which depends on the following lemma. The proof of this lemma follows from now-standard ideas introduced by C. Fefferman and Stein [9], and details may be found in this reference.

LEMMA 4.5. *Assume that the condition (S_p) holds. There exists a constant $C > 1$ such that if we define, for $x \in \partial B$,*

$$h(x) = \sup_{0 < h < \text{rad } \mathcal{A}_0} \{h : S_h(a)(x) < C\},$$

where

$$S_h(a)(x) = \left[\int_{\Gamma_h(x)} \frac{a^2(Y)}{\delta^n(Y)} dY \right]^{1/2},$$

then for any $y \in \partial B$, $|\{x \in \mathcal{A}(y, h) : h(x) > h\}| \geq ch^{n-1}$.

We are now ready to estimate $NF_2^0(Q)$.

PROPOSITION 4.6. $NF_2^0(Q) \leq C \{M[S'(u_0)](Q)\}^{1/p'}$.

Proof of (4.6). Let $Y \in \Omega_0$ and let $y \in A_0 \subseteq \partial B_1$, $t \in \mathbb{R}$. We can write $Y = (y, t)$ by taking Y to be the point on the ray joining the origin and y , of distance $(1 - t)$ from the origin. Then

$$|F_2^0(Z)| \lesssim \int_{y \in A_0} \int_{0 < t < \text{rad } A_0} |\varepsilon(y, t)| |\nabla_Y G_1(Z, y, t)| |\nabla u_0(y, t)| dt d\sigma(y)$$

which by Lemma 4.5 is

$$\begin{aligned} &\leq \frac{1}{C} \int_{y \in A_0} \int_{0 < t < \text{rad } A_0} \frac{1}{t^{n-1}} |\{x \in A(y, t) : h(x) > t\}| \\ &\quad \cdot |\varepsilon(y, t)| |\nabla_Y G_1(Z, y, t)| |\nabla u_0(y, t)| dt d\sigma(y) \\ &= \frac{1}{C} \int_{y \in A_0} \int_{0 < t < \text{rad } A_0} \frac{1}{t^{n-1}} \\ &\quad \cdot \left[\int_{\{x \in A(y, t) : h(x) > t\}} d\sigma(x) |\varepsilon(y, t)| |\nabla_Y G_1(Z, y, t)| |\nabla u_0(y, t)| dt d\sigma(y) \right] \end{aligned}$$

which by Fubini is

$$\begin{aligned} &\lesssim \int_{x \in 2A_0} \left[\iint_{Y \in \Gamma_{h(x)}(x)} t^{1-n} |\varepsilon(Y)| \right. \\ &\quad \cdot |\nabla_Y G_1(Z, Y)| |\nabla u_0(Y)| d\sigma(y) dt \Big] d\sigma(x) \\ &\leq C \int_{x \in 2A_0} \sum_{I \subseteq 2A_0} \left[\int_{Y \in \Gamma_{h(x)}^I(x)} t^{1-n} |\varepsilon(Y)| \right. \\ &\quad \cdot |\nabla_Y G_1(Z, Y)| |\nabla u_0(Y)| dY \Big] d\sigma(x), \end{aligned}$$

where $\Gamma_{h(x)}^I(x) = I^+ \cap \Gamma_{h(x)}(x)$. Now we use an argument entirely analogous to that used in the estimate of F_2^0 in Section 3 (using Cacciopoli and Lemma 3.6), to obtain the bound:

$$\begin{aligned} |F_2^0(Z)| &\lesssim \frac{1}{\omega_1(A)} \int_{x \in 2A_0} \left[\int_{Y \in \Gamma_{h(x)}(x)} \frac{a^2(Y)}{\delta^n(Y)} \left| \frac{G_1(Y)}{\delta(Y)} \right|^2 dY \right]^{1/2} Su_0(x) d\sigma(x) \\ &\lesssim \frac{1}{\omega_1(A)} \int_{x \in 2A_0} \left[\int_{Y \in \Gamma_{h(x)}(x)} \frac{a^2(Y)}{\delta^n(Y)} \left| \frac{G_1(Y)}{\delta(Y)} \right|^p dY \right]^{1/p} \\ &\quad \cdot \left[\int_{Y \in \Gamma_{h(x)}(x)} \frac{a^2(Y)}{\delta^n(Y)} dY \right]^{(p-2)/2p} Su_0(x) d\sigma(x) \end{aligned}$$

which by the definition of $h(x)$ in Lemma 4.5 is

$$\leq \frac{C}{\omega_1(\Delta)} \int_{x \in 2\Delta_0} \left[\int_{Y \in \Gamma_{h(x)}(x)} \frac{a^2(Y)}{\delta^n(Y)} \left| \frac{G_1(Y)}{\delta(Y)} \right|^p dY \right]^{1/p} Su_0(x) d\sigma(x)$$

Now, Hölder's inequality with exponent p , plus the doubling of σ and ω , allows us to apply the condition (S_p) to obtain the bound

$$\{M[S^{p'}(u_0)](Q_0)\}^{1/p'}.$$

Thus we have shown Proposition 4.6. ■

The remaining portions of F are:

$$F_2^j(Z) = \int_{R_j} \nabla_Y G_1(Z, Y) \cdot \varepsilon(Y) \nabla u_0(Y) dY$$

for $j = 1, \dots, N$.

For each $j = 1, \dots, N-1$, we further subdivide $R_j = (R_j \cap \Gamma_\gamma(Q_0)) \cup (R_j \setminus \Gamma_\gamma(Q_0))$, where the aperture γ is chosen small enough such that for any $Z \in \Gamma_\gamma(Q_0)$, $Q_0 \in \mathcal{A}(X^*, \delta(X)/2)$. Then, we break up F_2^j into

$$\begin{aligned} F_2^j(Z) &= \int_{R_j \cap \Gamma_\gamma(Q_0)} \nabla_Y G_1(Z, Y) \cdot \varepsilon(Y) \nabla u_0(Y) dY \\ &\quad + \int_{R_j \setminus \Gamma_\gamma(Q_0)} \nabla_Y G_1(Z, Y) \cdot \varepsilon(Y) \nabla u_0(Y) dY \\ &= F_3^j(Z) + F_4^j(Z). \end{aligned}$$

Consider the $F_3^j(Z)$ first. As in Section 3, let $R_j \cap \Gamma_\gamma(Q_0) = R_j^F$, $\Delta_j = \partial\Omega_j \cap \partial B_1$, and let X_j be a point in R_{j+1}^F such that $\delta(X_j) \approx 2^{j-1}\delta(X)$. Then let $Z = X$, our original fixed point in $\Gamma(Q_0)$. We proceed precisely as we did for F_3^j in the proof of Theorem 3.1. Namely, we use Cacciopoli's inequality and then apply Lemma 3.7 to move the pole of the Green's function from X to 0. We thus obtain

$$\begin{aligned} F_3^j(X) &\lesssim \frac{2^{-j\alpha} |\Delta_j|}{\omega_1(\Delta_j)} \left[\frac{1}{|\Delta_j|} \int_{\Omega_j} \left| \frac{G_1(Y)}{\delta(Y)} \right|^p \frac{a^2(Y)}{\delta(Y)} dY \right]^{1/p} \\ &\quad \cdot \left[\frac{1}{|\Delta_j|} \int_{R_j^F} \frac{a^2(Y)}{\delta(Y)} dY \right]^{(p-2)/2p} \cdot \left[\int_{R_j^F} |\nabla u_0(Y)|^2 \delta^{2-n}(Y) dY \right]^{1/2} \end{aligned}$$

Now apply the condition (S_p) , to obtain the bound

$$\frac{2^{-j\alpha}}{\omega_1(\Delta_j)} |\Delta_j| \left[C \frac{\omega_1(\Delta_j)}{|\Delta_j|} \right] (\|a\|_\infty)^{(p-2)/p} \left[\frac{1}{|\Delta_j|} \int_{R_j^F} \frac{dY}{\delta(Y)} \right]^{(p-2)/2p} \\ \cdot \left[\int_{R_j^F} |\nabla u_0(Y)|^2 \delta^{2-n}(Y) dY \right]^{1/2}$$

Now we note that for any of the $j=1, \dots, N$, any $Y \in R_j^F$ satisfies $\delta(Y) \approx r_j$ and $\text{vol}(R_j^F) \approx r_j^n$. Thus we get the above majorized by

$$C 2^{-j\alpha} \cdot \left[\int_{R_j^F} |\nabla u_0(Y)|^2 \delta^{2-n}(Y) dY \right]^{1/2} \leq C 2^{-j\alpha} S u_0(Q_0)$$

where C depends only on n (the dimension), p , $\|a\|_\infty$ and the ellipticity constants. Hence, $F_3^j(X)$ can be summed over j , to get $\sum_j F_3^j(X) \leq C S u_0(Q_0)$, where C is independent of the point X .

Next, to treat $F_4^j(X) = \int_{R_j \setminus \Gamma_\gamma(Q_0)} \nabla_Y G_1(X, Y) \cdot \varepsilon(Y) \nabla u_0(Y) dY$, $j=1, \dots, N$, we again segment $R_j \setminus \Gamma_\gamma(Q_0)$ into J sections each having length $\approx r_j$. In general, we can take $J \approx \sigma_{n-2}$ where σ_{n-2} = the surface measure of the $(n-2)$ -dimensional unit sphere.

Now note that $R_j \subset \Omega_j$ and break $\Omega_j \setminus \Gamma_\gamma(Q_0)$ into $K=1, \dots, J$ regions over Δ_j^K which are each essentially Carleson regions over surface balls of radius $\approx \frac{1}{2}r_j$. We let $\Omega_j \setminus \Gamma_\gamma(Q_0) = \bigcup_{K=1}^J \tilde{T}(\Delta_j^K)$, where $X \notin T(\Delta_j^K)$. Then

$$|F_4^j(X)| \leq \sum_{K=1}^J \int_{T(\Delta_j^K)} \nabla_Y G_1(X, Y) \cdot \varepsilon(Y) \nabla u_0(Y) dY$$

An argument as for the potential over Ω_0 gives

$$\int_{T(\Delta_j^K)} |\nabla_Y G_1(X, Y) \cdot \varepsilon(Y) \nabla u_0(Y)| dY \\ \lesssim \int_{x \in \Delta_j^K} \sum_{I \subseteq \Delta_j^K} \sup_{Y \in I_{h(x)}^I} |\varepsilon(Y)| l(I)^{1-n} \\ \cdot \left[\int_{Y \in I_{h(x)}^I} |\nabla_Y G_1(X, Y)|^2 dY \right]^{1/2} \\ \cdot \left[\int_{Y \in I_{h(x)}^I} |\nabla u_0(Y)|^2 dY \right]^{1/2} d\sigma(x)$$

After now applying Cacciopoli, we use Lemma 3.8, as all the arguments there apply to the case here. Thus, just as in Section 3,

$$\begin{aligned} & \int_{T(\mathcal{A}_j^K)} |\nabla_Y G_1(X, Y) \cdot \varepsilon(Y) \nabla u_0(Y)| dY \\ & \lesssim \frac{2^{-j\alpha}}{\omega_1(\mathcal{A}_j^K)} |\mathcal{A}_j^K| \left[\frac{1}{|\mathcal{A}_j^K|} \int_{Y \in \tilde{T}(\mathcal{A}_j^K)} \frac{a^2(Y)}{\delta(Y)} \left| \frac{G_1(Y)}{\delta(Y)} \right|^p dY \right]^{1/p} \\ & \quad \cdot \left[\frac{1}{|\mathcal{A}_j^K|} \int_{x \in \mathcal{A}_j^K} |Su_0(x)|^{p'} d\sigma(x) \right]^{1/p'}. \end{aligned}$$

Now the condition (S_p) plus doubling of surface measure, gives the bound

$$C 2^{-j\alpha} \{M[S^{p'}(u_0)](Q_0)\}^{1/p'}.$$

Thus, when we sum over $J = 1, \dots, K$ and $j = 1, \dots, N-1$, we get

$$\sum_j F_4^j(X) \leq C \{M[S^{p'}(u_0)](Q)\}^{1/p'},$$

where C does not depend on X .

Finally, we bound the last piece, $F_2^N(Z)$, the potential over $R_N = \Omega_N \setminus \Omega_{N-1}$. Note that $R_N \subseteq R$, where $R = B_1 \setminus B(Q_0, \frac{1}{2})$, so

$$F_2^N(X) \leq \int_R \nabla_Y G_1(X, Y) \cdot \varepsilon(Y) u_0(Y) dY.$$

As in Section 3, we need to bound

$$\int_{R \setminus B(0, 1/2)} \nabla_Y G_1(X, Y) \cdot \varepsilon(Y) u_0(Y) dY.$$

The same argument as for $F_x^N(Z)$ in Section 3 works here, given a variant of Lemma 3.8. The proof is entirely analogous to the proof of Lemma 3.8, and the reader may refer to the details there.

LEMMA 4.7. *For $Y \in R \setminus B(0, \frac{1}{2})$, if (S_p) holds, then for $\mathcal{A} \subseteq \partial B_1 \cap \partial R$, there is a constant C such that the following condition also holds:*

$$\left[\int_{T(\mathcal{A})} \frac{a^2(Y)}{\delta(Y)} \left(\frac{G_1(X, Y)}{\delta(Y)} \right)^p \frac{dY}{\sigma(\mathcal{A})} \right]^{1/p} \leq C \frac{\omega_1^X(\mathcal{A})}{\sigma(\mathcal{A})}.$$

To continue the estimate of the last piece, we break up the final region $R \setminus B(0, \frac{1}{2})$ into $k = 1, \dots, K$ Carleson type regions $T(\mathcal{A}^k)$ of length $\approx \frac{1}{2}$, where K depends only on the dimension n . Then, with $Y = (y, t)$ where $y \in \partial B_1$

and $t \in \mathbb{R}$, and all definitions as in the discussion of F_2^0 , we proceed as before and obtain:

$$\begin{aligned} & \int_{T(\mathcal{A}^k)} \nabla_Y G_1(X, Y) \cdot \varepsilon(Y) \nabla u_0(Y) dY \\ & \lesssim \int_{x \in \mathcal{A}^k} \left[\sum_{I \in \mathcal{A}^k} \int_{\Gamma_{h(x)}^I} \frac{a^2(Y)}{\delta^n(Y)} \left(\frac{G_1(X, Y)}{\delta(Y)} \right)^2 dY \right]^{1/2} Su_0(x) dx \\ & \lesssim \int_{x \in \mathcal{A}^k} \left[\int_{\Gamma_{h(x)}(x)} \frac{a^2(Y)}{\delta^n(Y)} \left(\frac{G_1(X, Y)}{\delta(Y)} \right)^p dY \right]^{1/p} \\ & \quad \cdot \left[\int_{\Gamma_{h(x)}(x)} \frac{a^2(Y)}{\delta^n(Y)} dY \right]^{(p-2)/2p} Su_0(x) d\sigma(x) \end{aligned}$$

which by the definition of $h(x)$ and Hölder's inequality is

$$\begin{aligned} & \leq C \left\{ \int_{x \in \mathcal{A}^k} \left[\int_{\Gamma_{h(x)}(x)} \frac{a^2(Y)}{\delta^n(Y)} \left(\frac{G_1(X, Y)}{\delta(Y)} \right)^p dY \right] d\sigma(x) \right\}^{1/p} \\ & \quad \cdot \left\{ \int_{x \in \mathcal{A}^k} [Su_0(x)]^{p'} d\sigma(x) \right\}^{1/p'}. \end{aligned}$$

Now Lemma 4.7 and the doubling of surface measure gives the bound $C\{M[S^{p'}(u_0)](Q_0)\}^{1/p'}$, and we are done.

Altogether, we have shown the estimate

$$\tilde{N}F(Q_0) \leq CSu_0(Q_0) + C[M(S^{p'}u_0)(Q_0)]^{1/p'},$$

which is the estimate (3.1.3). Hence, by arguments identical to those at the end of Section 3, we obtain the *a priori* estimate $\|Nu_1\|_{p'} \leq C\|g\|_{p'}$, and the proof of Theorem 3.2 is complete. ■

5. THE EQUIVALENCE OF THE TWO SOLVABILITY CONDITIONS

We now demonstrate that Theorems 3.1 and 3.2 are in fact equivalent in the case where $p \geq 2$; that is, the conditions (QS_p) and (S_p) are equivalent. This equivalence will be important in Section 6, where we use both conditions to show the sharpness of our results.

We remark that proving this equivalence of (3.1) and (3.2) does not make our work in Section 4 to prove Theorem 3.2 redundant, for the proof of equivalence will depend on Theorem 3.2 in a crucial way.

We now treat each direction of the equivalence separately.

THEOREM 5.1. *Take $p \geq 2$. For two elliptic operators L_0 and L_1 on $B_1 \subseteq \mathbb{R}^n$, the condition (S_p) implies that (QS_p) holds.*

Proof of (5.1). The key to this proof will be the fact that (S_p) implies that $k_1 \in B_p(d\sigma)$, by Theorem 3.2.

To show that (QS_p) holds, consider, for any $\Delta \subseteq \partial B_1$,

$$\int_{Q \in \Delta} \left[\int_{X \in \Gamma(Q)} \frac{a^2(X)}{\delta^n(X)} \left(\frac{G_1(X)}{\delta(X)} \right)^2 dX \right]^{p/2} \frac{d\sigma(Q)}{\sigma(\Delta)}.$$

By the comparison theorem (2.5), the above is equivalent to

$$\int_{Q \in \Delta} \left[\int_{X \in \Gamma(Q)} \frac{a^2(X)}{\delta^n(X)} \frac{\omega_1(\Delta_X)}{\sigma(\Delta_X)} \left(\frac{G_1(X)}{\delta(X)} \right) dX \right]^{p/2} \frac{d\sigma(Q)}{\sigma(\Delta)}.$$

Now let $\tilde{\Delta}$ be a three-fold enlargement of Δ_X , and recall the notation $d\omega_1 = k_1 d\sigma$. Note that then $Q \in \tilde{\Delta}$, so we have the relation:

$$\frac{\omega_1(\Delta_X)}{\sigma(\Delta_X)} \lesssim CM(k_1 \chi_{\tilde{\Delta}})(Q). \quad (5.1.1)$$

Define $\tilde{k}_1 = k_1 \chi_{\tilde{\Delta}}$.

Now we note that Theorem 3.2 gives us $k_1 \in B_p(d\sigma)$, and by Gehring's result ([13]), k_1 is also in $B_{p+\varepsilon}(d\sigma)$ for some small enough ε . We choose ε small enough so that if we let $q = p(p+\varepsilon)/(p+2\varepsilon)$ and $\tau = p - q$, then τ is small enough so that $(p-1)(p-\tau)/\tau > 2$. We will need this relation later. Note also that $p > q$; this is crucial to the argument.

Using the above estimate (5.1.1) plus Hölder's inequality with exponent $2(p+\varepsilon)/p$, we obtain the bound

$$\begin{aligned} & \frac{1}{\sigma(\Delta)} \left[\int_{Q \in \Delta} (M\tilde{k}_1(Q))^{p+\varepsilon} d\sigma(Q) \right]^{p/2(p+\varepsilon)} \\ & \cdot \left[\int_{Q \in \Delta} \left(\int_{X \in \Gamma(Q)} \frac{a^2(X)}{\delta^n(X)} \frac{G_1(X)}{\delta(X)} dX \right)^q d\sigma(Q) \right]^{(p+2\varepsilon)/2(p+\varepsilon)}. \end{aligned}$$

Recall (see [24]) that the maximal function M is strong type $(p+\varepsilon, p+\varepsilon)$ for $(p+\varepsilon) > 1$, and furthermore $k_1 \in B_{p+\varepsilon}(d\sigma)$. Hence the above is bounded by

$$C \left[\frac{\omega_1(\Delta)}{\sigma(\Delta)} \right]^{p/2} \cdot \left[\int_{Q \in \Delta} \left(\int_{X \in \Gamma(Q)} \frac{a^2(X)}{\delta^n(X)} \frac{G_1(X)}{\delta(X)} dX \right)^q dQ \right]^{(p+2\varepsilon)/2(p+\varepsilon)},$$

and by Hölder's with exponent p we obtain the bound

$$\left[\frac{\omega_1(\mathcal{A})}{\sigma(\mathcal{A})} \right]^{p/2} \cdot \left\{ \int_{Q \in \mathcal{A}} \left[\int_{X \in \Gamma(Q)} \frac{a^2(X)}{\delta^n(X)} \left(\frac{G_1(X)}{\delta(X)} \right)^p dX \right]^{q/p} \cdot \left[\int_{X \in \Gamma(Q)} \frac{a^2(X)}{\delta^n(X)} dX \right]^{q/p'} d\sigma(Q) \right\}^{(p+2\varepsilon)/2(p+\varepsilon)}.$$

Since $p/q > 1$, we can further bound the above by

$$\left[\frac{\omega_1(\mathcal{A})}{\sigma(\mathcal{A})} \right]^{p/2} \cdot \left\{ \left[\frac{1}{\sigma(\mathcal{A})} \int_{Q \in \mathcal{A}} \left[\int_{X \in \Gamma(Q)} \frac{a^2(X)}{\delta^n(X)} \left(\frac{G_1(X)}{\delta(X)} \right)^p dX \right] d\sigma(Q) \right]^{q/p} \cdot \left[\frac{1}{\sigma(\mathcal{A})} \int_{Q \in \mathcal{A}} \left[\int_{X \in \Gamma(Q)} \frac{a^2(X)}{\delta^n(X)} dX \right]^{(q/p')(p/(p-q))} d\sigma(Q) \right]^{(p-q)/p} \right\}^{(p+2\varepsilon)/2(p+\varepsilon)},$$

which by applying (S_p) is

$$\lesssim \left[\frac{\omega_1(\mathcal{A})}{\sigma(\mathcal{A})} \right]^{p/2} \cdot \left[\frac{\omega_1(\mathcal{A})}{\sigma(\mathcal{A})} \right]^{p/2} \cdot \left[\frac{1}{\sigma(\mathcal{A})} \int_{Q \in \mathcal{A}} \left[\int_{X \in \Gamma(Q)} \frac{a^2(X)}{\delta^n(X)} dX \right]^{(p-1)(p-\tau)/\tau} d\sigma(Q) \right]^{(p+2\varepsilon)/2(p+\varepsilon)}.$$

Recall that ε was chosen so that $(p-1)(p-\tau)/\tau > 2$. At this point, we recall that (S_p) implies that $(a^2(X)/\delta(X)) dX$ is a Carleson measure with respect to $d\sigma$. This Carleson measure property then implies that for any exponent $\alpha > 2$,

$$\int_{Q \in \mathcal{A}} \left[\int_{X \in \Gamma(Q)} \frac{a^2(X)}{\delta^n(X)} dX \right]^\alpha \frac{d\sigma(Q)}{\sigma(\mathcal{A})} \leq C.$$

Thus, (QS_p) holds, which completes the proof of Theorem 5.1. ■

THEOREM 5.2. *Take $p \geq 2$. For two elliptic operators L_0 and L_1 on $B_1 \subseteq \mathbb{R}^n$, the condition (QS_p) implies that (S_p) holds also.*

Proof of (5.2). To show that (S_p) holds, we look at

$$\left[\int_{X \in \Gamma(\mathcal{A})} \frac{a^2(X)}{\delta(X)} \left(\frac{G_1(X)}{\delta(X)} \right)^p \frac{dX}{\sigma(\mathcal{A})} \right].$$

By the comparison theorem together with the relations (5.1.1), we have the above bounded by a constant multiple of

$$\begin{aligned} & \int_{Q \in \mathcal{A}} \int_{X \in \Gamma(Q)} \frac{a^2(X)}{\delta^n(X)} \left(\frac{G_1(X)}{\delta(X)} \right)^2 [M(k_1)(Q)]^{p-2} \frac{dX}{\sigma(\mathcal{A})} d\sigma(Q) \\ & \leq \left\{ \int_{Q \in \mathcal{A}} \left[\int_{X \in \Gamma(Q)} \frac{a^2(X)}{\delta^n(X)} \left(\frac{G_1(X)}{\delta(X)} \right)^2 \frac{dX}{\sigma(\mathcal{A})} \right]^{p/2} d\sigma(Q) \right\}^{2/p} \\ & \quad \cdot \left\{ \int_{Q \in \mathcal{A}} [M(k_1)(Q)]^p \frac{dX}{\sigma(\mathcal{A})} \right\}^{(p-2)/p}. \end{aligned}$$

Finally, we use the fact that M is strong type (p, p) for $p > 1$, together with the fact that $k_1 \in B_p$ by Theorem 3.1, to get the bound

$$\left\{ \int_{Q \in \mathcal{A}} \left[\int_{X \in \Gamma(Q)} \frac{a^2(X)}{\delta^n(X)} \left(\frac{G_1(X)}{\delta(X)} \right)^2 \frac{dX}{\sigma(\mathcal{A})} \right]^{p/2} d\sigma(Q) \right\}^{2/p} \cdot \left(\frac{\omega_1(\mathcal{A})}{\sigma(\mathcal{A})} \right)^{p-2}.$$

Applying the assumption that (QS_p) holds gives the bound $C(\omega_1(\mathcal{A})/\sigma(\mathcal{A}))^p$ and completes the argument for Theorem 5.2. ■

6. SHARPNESS OF THE MAIN RESULTS

In this section, Δ will denote the Laplacian. Our goal now is to show that the condition in Theorem 3.1 (resp. 3.2) is sharp in the following sense: given the hypothesis of condition (QS_p) (resp. (S_p)), the conclusion that $k_1 \in B_p(dx)$ is the strongest possible.

To show this sharpness, our tools will be certain operators L_1 constructed from doubling weights f ; L_0 will be the Laplacian. We will work in \mathbb{R}_+^2 , following ideas in [12] which stem from the Beurling–Ahlfors constructions in [1] and [2]. We construct a quasiconformal map $\Phi: \mathbb{R}_+^2 \rightarrow \mathbb{R}_+^2$, and take L_1 to be the pullback of Δ under Φ . That is, we will let $L_1 = \operatorname{div} A \nabla$, where $A = (\det \Phi') \cdot ((\Phi')^{-1})^t \cdot (\Phi')^{-1}$, where (Φ') is the matrix of derivatives.

Specifically, given any doubling weight f , let $F(x) = \int_0^x f(t) dt$, $\phi(x) = (1/\sqrt{\pi}) e^{-x^2}$, $\psi(x) = -\phi'(x)$. Set $\Phi(x, t) = (F * \phi_t(x), F * \psi_t(x)) = (\Phi_1(x, t), \Phi_2(x, t))$, where $\phi_t(x) = t^{-1} \phi(x/t)$. At this point, we cite Lemma 4.4 of [12], which states that Φ is quasiconformal and has trace F ; that is, $\det \Phi'(x, t) \gtrsim |\nabla \Phi(x, t)|^2$ and $\Phi(x, 0) = (F(x), 0)$. Although this lemma is stated for $f \in A_\infty$, in fact the proof uses only the fact that f is doubling.

Then L_1 is defined as above. Note that for the above L_1 , $f \, dx \approx \omega_1$ on compact subsets of \mathbb{R} , as shown in [2] and in [12]. That is, for all compact K in \mathbb{R} , there exist constants c_1 and c_2 such that

$$c_1 \omega_1(E) \leq \int_E f \, dx \leq c_2 \omega_1(E),$$

where E is any measurable subset of K .

Given the above construction, the sharpness of both Theorems 3.1 and 3.2 will be direct consequences of the following equivalence in the quasiconformal setting.

THEOREM 6.1. *Let $p > 1$. Let f be a doubling weight on $\mathbb{R} = \partial\mathbb{R}_+^2$. Take $L_0 = \Delta$ and $L_1 = \operatorname{div} A \nabla$ as constructed above, with $d\omega_1 = k_1 \, dx$. Then $f \in B_p(dx)$ if and only if (S_p) holds for L_0 and L_1 .*

COROLLARY 6.2. *Theorem 3.2 is sharp in the following sense: Let $L_0 = \Delta$ in \mathbb{R}_+^2 , and suppose f is an arbitrary weight in $B_p(dx)$ on $\mathbb{R} = \partial\mathbb{R}_+^2$. Then there exists an elliptic operator L_1 in \mathbb{R}_+^2 with bounded measurable coefficients such that (S_p) holds, and such that $\omega_1 \approx f \, dx$ on compact subsets of \mathbb{R} .*

Corollary 6.2 says that, in the two dimensional case, as we range through all perturbations of Δ that satisfy the condition of Theorem 3.2, we will range through all possible B_p weights. So the condition (S_p) cannot guarantee anything stronger than preservation of the class B_p .

COROLLARY 6.3. *Theorem 3.1 is sharp in the sense described in Corollary 6.2.*

Given Corollary 6.2, the sharpness of Theorem 3.1 follows immediately from the proof of Theorem 5.1. For, in this special quasiconformal setting, an arbitrary $f \in B_p(dx)$, $p > 1$ gives rise to an L_1 which satisfies condition (S_p) , so that it also satisfies (QS_p) , by the exact argument used to prove Theorem 5.1. Thus Theorem 3.1 also states the strongest possible conclusion from its hypothesis.

It remains to prove Theorem 6.1. We first rewrite the condition (S_p) for this quasiconformal setting, according to the following lemma.

LEMMA 6.4. *For the \mathbb{R}_+^2 operators $L_0 = \Delta$ and L_1 as constructed above, the condition (S_p) of Theorem 3.2 is equivalent to the following condition holding for all $x_0 \in \mathbb{R}$:*

$$\frac{1}{t} \int_{s=0}^t \int_{|x-x_0| < t} a^2(x, s) |f * \phi_s(x)|^p \frac{dx \, ds}{s} \leq C \left[\frac{1}{t} \int_{|x-x_0| < t} f \, dx \right]^p \quad (\tilde{S}_p)$$

where $\phi(x) = 1/\sqrt{\pi} e^{-x^2}$ and $\phi_s(x) = (1/s) \phi(x/s)$, and where f is the doubling weight from which L_1 was constructed.

Proof of (6.4). Using the fact that $f dx \approx k_1 dx$, in two dimensions (S_p) can be written

$$\frac{1}{t} \int_{s=0}^t \int_{|x-x_0| < t} \frac{a^2(x, s)}{s} \left[\frac{G_1(0, (x, s))}{s} \right]^p dx ds \leq C \left[\frac{1}{t} \int_{|x-x_0| < t} f dx \right]^p.$$

So we need to show that

$$|f * \phi_s(x)| \approx \frac{G_1(0, (x, s))}{s} = \frac{G_1((x, s))}{s}$$

for $0 < s < t$ and $|x - x_0| < t$.

We show the \lesssim direction first; the \gtrsim direction is easier. Note that

$$\begin{aligned} \sqrt{\pi}(f * \phi_s(x)) &= \int_{|x-y| < s} f(y) \frac{1}{s} \phi\left(\frac{x-y}{s}\right) dy \\ &\quad + \int_{|x-y| > s} f(y) \frac{1}{s} \phi\left(\frac{x-y}{s}\right) dy. \end{aligned}$$

Now,

$$\begin{aligned} \int_{|x-y| < s} f(y) \frac{1}{s} \phi\left(\frac{x-y}{s}\right) dy &\lesssim \int_{|x-y| < s} f(y) \frac{1}{s} e^{-((x-y)/s)^2} dy \\ &\leq \int_{|x-y| < s} f(y) \frac{1}{s} dy \\ &\approx \frac{1}{s} \omega_1\{y: |x-y| < s\} \end{aligned}$$

At this point we use the comparison theorem (2.5), so that the above is comparable to

$$\frac{G_1((x, s))}{s}.$$

We now bound the integral over $|x - y| > s$ by summing over dyadic regions and using the doubling of f .

$$\begin{aligned} & \sum_{j=0}^{\infty} \int_{2^j s < |x-y| < 2^{j+1} s} f(y) \frac{1}{s} \frac{1}{\sqrt{\pi}} e^{-((x-y)/s)^2} dy \\ & \lesssim \sum_{j=0}^{\infty} e^{-16^j} \frac{1}{s} \rho^{j+1} \omega_1\{y: |x-y| < s\}, \end{aligned}$$

where ρ is the doubling constant for the measure ω_1 . Finally, by (2.5) again, the above is bounded by

$$\frac{G_1((x, s))}{s} \sum_{j=0}^{\infty} e^{-16^j} \rho^{j+1} \lesssim \frac{G_1((x, s))}{s}.$$

To see the \gtrsim direction, we need only note that

$$\begin{aligned} |f * \phi_s(x)| & \gtrsim \int_{|x-y| < s} f(y) \frac{1}{s} e^{-((x-y)/s)^2} dy \\ & \approx \frac{G_1((x, s))}{s}. \end{aligned}$$

Thus Lemma 6.4 is complete. ■

We will use the condition (\tilde{S}_p) from Lemma 6.4 as our characterization of (S_p) in proving (6.1). In addition, we will make use of the following characterization of B_p , which was introduced by R. Fefferman, Kenig and Pipher in [12]. Define the class of approximate identities $\mathcal{A}_{N_0} = \{\varphi \in \mathcal{S}(\mathbb{R}^n) : \int |D^\alpha \varphi(x)|^2 (1 + |x|)^{N_0} \leq 1, |\alpha| \leq N_0\}$, where \mathcal{S} denotes the Schwartz functions. Then we have the following.

THEOREM 6.5 [12]. *Suppose w is a doubling weight on $\mathbb{R}^n(dx)$, with doubling constant ρ . Suppose $\varphi \in \mathcal{A}_{N_0}$, $N_0 = N_0(\rho)$, $\int \varphi = 1$, and $\psi = \nabla \varphi$. Then $w \in B_p(dx)$ if and only if, for all $x_0 \in \mathbb{R}^n$ and all $t > 0$,*

$$\begin{aligned} & \left[\frac{1}{t^n} \int_{s=0}^t \int_{|x-x_0| < t} \frac{|w * \psi_s(x)|^2}{|w * \varphi_s(x)|^2} |w * \varphi_s(x)|^p dx \frac{ds}{s} \right]^{1/p} \\ & \leq C \left[\frac{1}{t^n} \int_{|x-x_0| < t} w(x) dx \right]. \end{aligned} \quad (6.5.1)$$

We will need one more lemma from [12] which will help to handle $a^2(x, t)$.

LEMMA 6.6 [12]. *Take all notation as above, and also let $A = (a_{ij})$, and set $\tilde{\phi}(x) = x\phi(x)$ and $\tilde{\psi}(x) = x\psi(x)$. Then the following relations hold:*

$$a_{11}(x, t) = [(f * \tilde{\psi}_t(x))^2 + (f * \psi_t(x))^2]/D$$

$$a_{12}(x, t) = a_{12}(x, t)$$

$$= -[(f * \tilde{\psi}_t(x))(f * \tilde{\phi}_t(x)) + (f * \psi_t(x))(f * \phi_t(x))]/D$$

$$a_{22}(x, t) = [(f * \tilde{\phi}_t(x))^2 + (f * \phi_t(x))^2]/D$$

where $D = (f * \phi_t(x))(f * \tilde{\psi}_t(x)) - (f * \psi_t(x))(f * \tilde{\phi}_t(x))$. Also, we have

$$|a_{11}(x, t) - 1| = \mathcal{O}\left(\frac{|f * \psi_t(x)|}{|f * \phi_t(x)|}\right) + \mathcal{O}\left(\frac{|f * \phi_t''(x)|}{|f * \phi_t(x)|}\right)$$

$$|a_{22}(x, t) - 1| = \mathcal{O}\left(\frac{|f * \psi_t(x)|}{|f * \phi_t(x)|}\right) + \mathcal{O}\left(\frac{|f * \phi_t''(x)|}{|f * \phi_t(x)|}\right)$$

$$|a_{12}(x, t)| = |a_{21}(x, t)| = \mathcal{O}\left(\frac{|f * \psi_t(x)|}{|f * \phi_t(x)|}\right).$$

Finally, since $\phi(x)$ is the heat kernel, we will be using the following result of Moser for heat solutions.

THEOREM 6.7 [21]. *If u is a solution to the heat equation in a parabolic rectangle $R = I \times J$, then*

$$\sup_{Z \in R'} |u(Z)| \leq C \left(\frac{1}{|R|} \int_R u^2(Z) dZ \right)^{1/2}$$

where $R' = I' \times J'$ is a subrectangle of R such that I and I' are concentric, and J, J' have the same right endpoint, and such that $|I'|/|I| = 1 - \varepsilon_1$, $|J'|/|J| = 1 - \varepsilon_2$. The constant C depends only on the parabolic constant for R , and on ε_1 and ε_2 .

We now prove Theorem 6.1.

Proof of (6.1). First, we demonstrate the forward direction. We have L_1 as constructed above from $f \in B_p(dx)$. We wish to show that (\tilde{S}_p) holds, which by Lemma 6.4 will mean (S_p) holds. By (6.6), we know

$$a^2(x, t) = \mathcal{O}\left(\frac{|f * \psi_t(x)|^2}{|f * \phi_t(x)|^2}\right) + \mathcal{O}\left(\frac{|f * \phi_t''(x)|^2}{|f * \phi_t(x)|^2}\right).$$

So

$$\begin{aligned}
& \frac{1}{T} \int_{s=0}^T \int_{|x-x_0| < T} a^2(x) |f * \phi_s(x)|^p \frac{dx ds}{s} \\
& \approx \frac{1}{T} \int_{s=0}^T \int_{|x-x_0| < T} \sup_{\substack{|x-y| < s/2 \\ s/2 < t < 3s/2}} \frac{|f * \phi_t''(y)|^2}{|f * \phi_t(y)|^2} |f * \phi_s(x)|^p \frac{dx ds}{s} \\
& \quad + \frac{1}{T} \int_{s=0}^T \int_{|x-x_0| < T} \sup_{\substack{|x-y| < s/2 \\ s/2 < t < 3s/2}} \frac{|f * \psi_t(y)|^2}{|f * \phi_t(y)|^2} |f * \phi_s(x)|^p \frac{dx ds}{s} \\
& = \mathbf{I}_3 + \mathbf{I}_4.
\end{aligned}$$

Consider \mathbf{I}_3 first. By a calculation in the proof of Theorem 4.2 of [12], we have the estimate that for $0 < s < T$, $|x - x_0| < T$,

$$\sup_{\substack{|x-y| < s/2 \\ s/2 < t < 3s/2}} \frac{|f * \phi_t''(y)|^2}{|f * \phi_t(y)|^2} \leq \frac{1}{s^2} \int_{\substack{|x-y| < s \\ s/4 < t < 9s/4}} \frac{|f * \phi_t''(y)|^2}{|f * \phi_t(y)|^2} dy dt.$$

The above estimate follows by applying Theorem 6.7 to the heat solution $f * \phi_{\sqrt{t}}(x)$ and using the doubling of f . Details may be seen in [12].

Thus, using this estimate, we have

$$\begin{aligned}
\mathbf{I}_3 & \leq \frac{1}{T} \int_{s=0}^T \int_{|x-x_0| < T} \frac{1}{s^2} \iint_{\substack{|x-y| < s \\ s/4 < t < (9s)/4}} \frac{|f * \phi_t''(y)|^2}{|f * \phi_t(y)|^2} \\
& \quad \times |f * \phi_s(x)|^p dy dt \frac{dx ds}{s}.
\end{aligned}$$

At this point, since $s \approx t$ and $|x - y| < s/2 \approx t/2$, we have $|f * \phi_s(x)| \approx |f * \phi_t(y)|$. This follows by an argument as in Lemma 6.4: break up the integral into dyadic regions and use the doubling of f . Thus, this equivalence $|f * \phi_s(x)| \approx |f * \phi_t(y)|$ gives us

$$\begin{aligned}
\mathbf{I}_3 & \leq \frac{1}{T} \int_{s=0}^T \int_{|x-x_0| < T} \frac{1}{s^2} \left[\iint_{\substack{|x-y| < s \\ s/4 < t < 9s/4}} \frac{|f * \phi_t''(y)|^2}{|f * \phi_t(y)|^2} \right. \\
& \quad \left. \times |f * \phi_t(y)|^p dy dt \right] \frac{dx ds}{s}.
\end{aligned}$$

Now we change the order of integration; letting K be some large constant and integrating in the variables x and s , we obtain the bound

$$C \frac{1}{T} \int_{t=0}^{KT} \int_{|y-x_0| < KT} \frac{|f * \phi_t''(y)|^2}{|f * \phi_t(y)|^2} |f * \phi_t(y)|^p \frac{1}{t} dy dt$$

which by the characterization of B_p in Theorem 6.5 is

$$\leq C \left[\frac{1}{T} \int_{|x-x_0| < T} f(x) dx \right]^p.$$

Thus, (\tilde{S}_p) holds, and we have the desired bound for \mathbf{I}_3 .

For \mathbf{I}_4 , a similar argument applies.

$$\begin{aligned} \mathbf{I}_4 &= \frac{1}{T} \int_{s=0}^T \int_{|x-x_0| < T} \sup_{\substack{|x-y| < s/2 \\ s/2 < t < (3s)/2}} \frac{|f * \psi_t(y)|^2}{|f * \phi_t(y)|^2} |f * \phi_s(x)|^p \frac{dx ds}{s} \\ &\leq \frac{1}{T} \int_{s=0}^T \int_{|x-x_0| < T} \frac{1}{s^2} \iint_{\substack{|x-y| < s \\ s/4 < t < (9s)/4}} \frac{|f * \phi_t'(y)|^2}{|f * \phi_t(y)|^2} \\ &\quad \times |f * \phi_s(x)|^p dy dt \frac{dx ds}{s}. \end{aligned}$$

To obtain the above inequality on the supremum, note that $f * \phi'_{\sqrt{t}}(x) = \sqrt{t}(\partial/\partial x) f * \phi_{\sqrt{t}}(x)$, and apply Moser's estimate for heat solutions to $(1/\sqrt{t}) f * \phi'_{\sqrt{t}}(x)$, and change variables as we did for \mathbf{I}_3 . Now, as in the argument for \mathbf{I}_3 , the above is bounded by

$$\begin{aligned} &\frac{1}{T} \int_{s=0}^T \int_{|x-x_0| < T} \frac{1}{s^2} \\ &\quad \times \left[\iint_{\substack{|x-y| < s \\ s/4 < t < (9s)/4}} \frac{|f * \phi_t'(y)|^2}{|f * \phi_t(y)|^2} |f * \phi_t(y)|^p dy dt \right] \frac{dx ds}{s}, \end{aligned}$$

and by once again interchanging the order of integration, we obtain the bound

$$\begin{aligned} &\frac{1}{T} \int_{t=0}^{KT} \int_{|y-x_0| < KT} \frac{|f * \phi_t'(y)|^2}{|f * \phi_t(y)|^2} |f * \phi_t(y)|^p \frac{1}{t} dy dt \\ &\leq C \left[\frac{1}{T} \int_{|x-x_0| < T} k_1(x) dx \right]^p. \end{aligned}$$

Thus, (S_p) is satisfied, which proves the forward implication of Theorem 6.1.

We now prove the reverse direction of Theorem 6.1. We assume that (\tilde{S}_p) holds, where $p > 1$, and we want to show that f satisfies (6.5.1), the characterization of $B_p(dx)$ given in [12]. Note that Theorem 3.2 gives us this fact directly for $p \geq 2$, but we still need a proof when $1 < p < 2$. Thus, we need to show that

$$a^2(x, t) \geq C \frac{|f * \phi'_t(x)|^2}{|f * \phi_t(x)|^2}$$

over the region of integration $\{(x, t): 0 < t < T, |x - x_0| < T\}$.

Let $A = (a_{ij})$, and let $X = (x, t)$. Then

$$a^2(X) = \sup_{Y \in B(X, \delta(X)/2)} |A(Y) - I|^2 = \sup_{|Z|=1} (A(x, t) - I)Z|^2.$$

Take $Z = (1, 0)$. Then we have the lower bound $|(a_{11}^2 + a_{21}^2)^{1/2}|^2 \geq a_{21}^2$. Now Lemma 6.6 gives us representations for a_{21} . Recall our notation $\tilde{\phi} = x\phi$ and $\tilde{\psi} = x\psi$. Letting $D = (f * \phi_t(x))(f * (x\psi)_t(x)) - (f * \psi_t(x))(f * (x\phi)_t(x))$, we have

$$\begin{aligned} a_{21}^2 &= \left| \frac{(f * (\tilde{\psi})_t(x))(f * (\tilde{\phi})_t(x)) + (f * \psi_t(x))(f * \phi_t(x))}{D} \right|^2 \\ &\approx \left| \frac{(f * (\tilde{\psi})_t(x))(f * (\tilde{\phi})_t(x)) + (f * \psi_t(x))(f * \phi_t(x))}{(f * \phi_t(x))^2} \right|^2, \end{aligned}$$

by quasiconformality of Φ and a computation to show $|\nabla\Phi(x, t)|^2 \approx |f * \phi_t(x)|^2$.

Since $\psi(x) \approx x\phi(x)$, the above will now be bounded from below by a multiple of $|f * \psi_t(x)/f * \phi_t(x)|^2$, as desired, once we can show the estimate $|f * \phi_t(x)| \leq C |f * \tilde{\psi}_t(x)|$. By translation and dilation invariance, it is enough to show that

$$|f * \phi(0)| \leq C |f * \tilde{\psi}(0)|.$$

The above inequality follows from the doubling of f and the rapid decay of ϕ , similar to the argument for Lemma 6.4.

Hence f satisfies the condition of Theorem 6.5, and thus is in $B_p(dx)$, so $k_1 \in B_p$ also. Theorem 6.1 is proved. ■

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